MINIMUM AREA POLYOMINO VENN DIAGRAMS

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Abstract. Polyomino Venn (or polyVenn) diagrams are Venn diagrams whose curves are the perimeters of orthogonal polyominoes drawn on the integer lattice. Minimum area polyVenn diagrams are those in which each of the 2^n regions, in a diagram of n polyominoes, consists of exactly one unit square.

We construct minimum area polyVenn diagrams in bounding rectangles of size $2^r \times 2^c$ whenever $r, c \geq 2$. Our construction is inductive, and depends on two "expansion" results. First, a minimum area polyVenn diagram in a $2^r \times 2^c$ rectangle can be expanded to produce another that fits into a $2^{r+1} \times 2^{c+1}$ rectangle. Secondly, when $r = 2$, it can also be expanded to produce a polyVenn diagram in a $2^r \times 2^{c+3}$ bounding rectangle. Finally, we construct polyVenn diagrams in bounding rectangles of size $(2^{n/2} - 1) \times (2^{n/2} + 1)$ if n is even, but where the empty set is *not* represented as a unit square.

1. Introduction and motivation

Consider the set of three L shaped tetrominoes A, B, C, shown at the top of Figure 1. If these are overlapped in the obvious way, then all subsets of {A,B,C} occur as a unique unit square in the result, the 2×4 rectangle shown at the bottom of Figure 1. In other words, the curves that comprise the perimeters of these tetrominoes form a Venn diagram when overlaid as shown.

Definition 1.1. A *polyomino* is an edge-connected set of unit squares, combined in such a way so there are no holes, thus allowing the perimeter to be a simple, closed curve. Some polyominoes are named by the number of unit squares they contain. For example, polyominoes containing 2 to 5 unit squares are respectively called dominoes, triominoes, tetrominoes or pentominoes.

We are using the term *Venn diagram* in this paper in the sense defined by Grünbaum in [3].

Definition 1.2. A n-Venn diagram consists of n simple closed curves $C_1, C_2, \cdots, C_n$ drawn in the plane such that each of the 2^n intersections

$$X_1 \cap X_2 \cap \cdots \cap X_n, \tag{1.1}$$

where X_i is either the open exterior or is the open interior of the curve C_i, is both non-empty and connected.

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The problem of finding and overlapping congruent polyominoes to simulate a Venn diagram was introduced by Thomson [5] on his website, where he displayed his solutions for up to 4 polyominoes. Chow and Ruskey [2] investigated the more general question of minimizing the number of unit squares occupied by the overlapping pentomino pieces and produced minimum area polyVenn diagrams for $n = 5, 6, 7$. It seems that minimum area polyVenn diagrams may find some use in data visualization since they have the nice feature that the same amount of area is available in each region for attaching labels. As an example, a polyVenn diagram from [2] is used for data visualization in a Microbiology journal [1].

The paper [2] also initiated the question of polyominoes being placed into a rectangle so that the resulting configuration is a Venn diagram in which each region is a single unit square. In this paper, we settle several of conjectures/questions from [2] with respect to these minimum area Venn diagrams that use polyominoes, here further referred to as *polyVenns* or *polyVenn diagrams*. In particular, we show that there is a minimum area polyVenn diagram for all $n \geq 1$, where n is the number of polyominoes. Furthermore, these diagrams are contained in bounding rectangles of various aspect ratios.

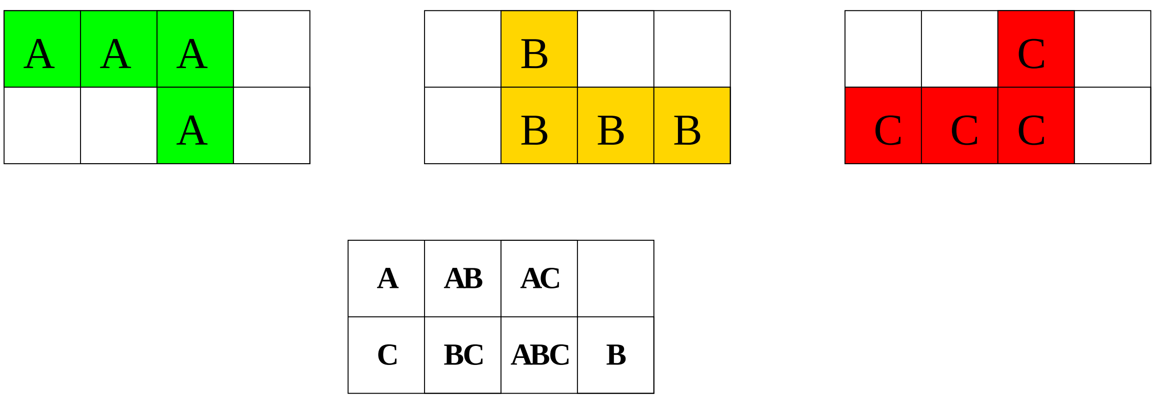


FIGURE 1. A $(1, 2)$-polyVenn.

Definition 1.3. A (r, c)-polyVenn is a minimum area polyVenn diagram formed by $r + c$ curves, each of which is the perimeter of a polyomino, placed into a $2^r \times 2^c$ rectangular grid.

For example, Figure 1 shows a $(1, 2)$-polyVenn. Note that in an n-set minimum area polyVenn, each polyomino covers exactly 2^{n-1} unit squares, and intersects with every other polyomino in exactly 2^{n-2} squares.

Proposition 1.4. *A $(0, c)$-polyVenn does not exist for any $c > 2$.*

Proof. Consider a $(0, c)$-polyVenn where $c \geq 2$. There are c polyomino pieces covering 2^c grids numbered $1, \ldots, 2^c$. Let the leftmost piece be P_1. Assume, without loss of generality, that the one uncovered grid square is on the left. Then, P_1 covers consecutive squares $2, \ldots, 2^{c-1}+1$. Let the rightmost piece be P_2. Then P_2 covers consecutive squares $2^{c-1}+1, \ldots, 2^c$. Thus the overlap of P_1 and P_2 is exactly one square. However, the number of overlapped squares for any two pieces must be 2^{c-2}, so we must have $2^{c-2} = 1$, that is $c = 2$. $\square$

2. Expanding an Existing Diagram

It is natural to ask if polyVenns exist for all n-sets. And if they do, is there a method to construct them? A well-known question [4] and an open problem [6], asks if a simple Venn diagram of n curves can be extended to a simple Venn diagram of $n+1$ curves. We ask a similar question. We want to know if it is possible to extend a polyVenn by enlarging the grid and adding more polyomino pieces.

We use a technique is called *expansion*. The idea is to take an existing (r, c)-polyVenn V and expand it into a $(r+r', c+c')$-polyVenn by the addition of $r'+c'$ polyominoes. The original polyominoes in V are "expanded" by uniformly stretching vertically by a factor of $2^{r'}$ and horizontally by a factor of $2^{c'}$.

The key element here is finding the $r'+c'$ new polyominoes. The construction of these new polyominoes will depend on r, r', c, c', but not otherwise on V. Figure 2 shows an expansion of the $(1, 2)$-polyVenn of Figure 1 into a $(2, 3)$-polyVenn.

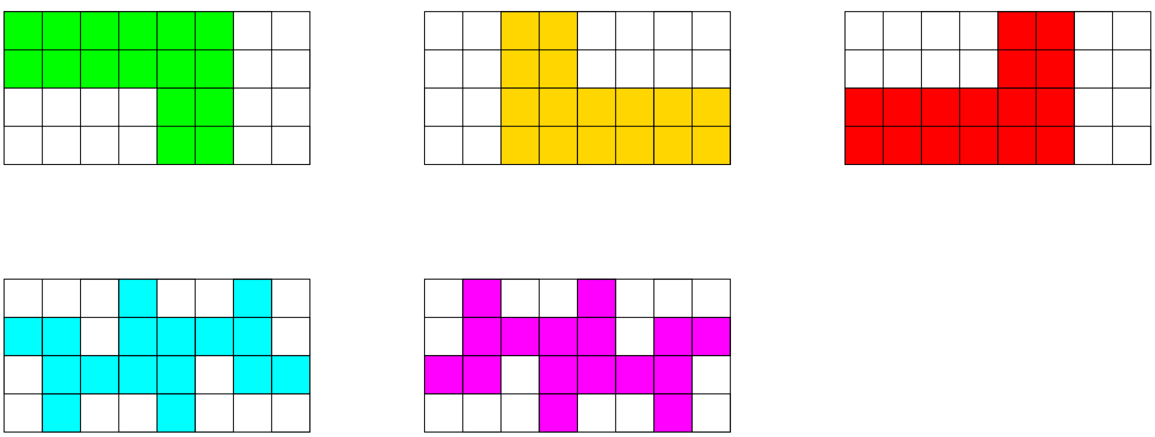


Figure 2. A $(2, 3)$-polyVenn created by expansion.

In the example, we create 2 new polyominoes: we place a single domino in each 2×2 expansion of the original unit squares of the of the $(1, 2)$-polyVenn. For each expanded 2×2 region of these two pieces, the dominoes form a $(1, 1)$-polyVenn. However, it is not generally necessary that each expansion contain a smaller polyVenn.

In the construction of a $2^{r'} \times 2^{c'}$ expansion, we are not limited to only connected pieces on the (r', c')-polyVenns. The pieces can be disconnected as long as they are connected on the larger grid. We start by describing the disconnected polyomino sets in terms of a *half-set system* (HSS).

Definition 2.1. An n-HSS is a collection $\{S_1, S_2, \ldots, S_n\}$ of subsets of $\{1, 2, \ldots, 2^n\}$ with the property that for any nonempty subset $A \subseteq \{1, 2, \ldots, n\}$

$$\Big|\bigcap_{i\in A} S_i\Big| = 2^{n-|A|}.$$

The following theorem provides an alternate way of checking for the HSS property.

Theorem 2.2. *Let $\{S_1, S_2, \ldots, S_n\}$ be a collection of subsets of $\{1, 2, \ldots, 2^n\}$. For all $A \subseteq \{1, 2, \ldots, n\}$*

$$\Big|\bigcap_{i\in A} S_i\Big| = 2^{n-|A|} \tag{2.1}$$

if and only if for all subsets $B \subseteq \{1, 2, \ldots n\}$, there is a unique element $m \in \{1, 2, \ldots, 2^n\}$ such that

$$\{m\} = \left(\bigcap_{i\in B} S_i\right) \cap \left(\bigcap_{i\notin B} \overline{S}_i\right). \tag{2.2}$$

Note that condition (2.2) is analogous to the Venn diagram condition that (1.1) is non-empty. Connectedness is trivial for polyVenns since each (1.1) intersection is a square. Thus the theorem provides us with another way of checking the Venn diagram conditions. This will prove useful not only for proving that the constructions are correct, but also computationally, particularly when the curves are represented using computer words, so that the intersections can be carried out in constant time. The proof of this Theorem is somewhat long and technical, so we put the proof in the Appendix; see Subsection 7.1 for the proof. A similar remark applies the the lemma in the following subsection.

We are interested in an HSS where the elements are unit squares on a grid. Using these, we can define the conditions that will allow for an expansion of a polyVenn diagram.

2.1. A sufficient condition.

Lemma 2.3. *Let G be a set of $2^r \times 2^c$ unit squares on a grid, each labeled $\{g_{ij} \mid 0 \leq i < 2^r, 0 \leq j < 2^c\}$ Let $X(G)$ be a $2^{r'} \times 2^{c'}$ expansion of G, where each g_{ij} in G expands to G_{ij}. Each G_{ij} is a $2^{r'} \times 2^{c'}$ mini-grid in $X(G)$. Let $P = \{P_1, P_2, \ldots, P_{r+c}\}$ be an HSS on G. Let $X(P)$ be a set of subsets, $\{X(P_1), X(P_2), \ldots, X(P_{r+c})\}$ on $X(G)$ where $X(P_j)$ has the following property:*

$$\text{If } g_{ij} \in P_j, \text{ then } G_{ij} \in X(P_j). \tag{2.3}$$

For every G_{ij} on $X(G)$, let $M_{ij} = \{M_{ij1}, M_{ij2}, \ldots, M_{ij(r'+c')}\}$ be an HSS on G_{ij}. Let E be a set of subsets on $X(G)$, $E = \{E_1, E_2, \ldots, E_{r'+c'}\}$ with the conditions that for *all* $k \in \{1, 2, \ldots, r' + c'\}$,

$$E_k = \bigcup_{\substack{0 \le i < 2^r \\ 0 \le j < 2^c}} M_{ijk}.$$

Then $X(P) \cup E$ *is an HSS on* $X(G)$.

See the Appendix subsection 7.2 for the proof.

2.2. **PolyVenns and HSSs.** It is easy to see that the set of polyomino pieces in a polyVenn, when represented as a set of $(r + c)$ subsets on the labeled unit squares of a $2^r \times 2^c$ grid, is an HSS. This HSS has the property that the set of unit squares of subset P_i must be connected and have no holes, i.e. P_i must be *simply connected*.

As with the complement of a set, the complement of P_i is the set of unit squares that are not contained in P_i. If $\overline{P_i}$ is also simply-connected, we say P_i is *self-complementing*. As it is done in Lemma 7.1, we can substitute any self-complementing polyomino piece with its complement and still have a polyVenn.

Now we have another way to define a polyVenn besides Definition 1.3.

Definition 2.4. A (r, c)-polyVenn is an HSS on the unit squares of a $2^r \times 2^c$ grid where each subset P_i, where $1 \le i \le r + c$, consists of a simply connected set of unit squares.

We use this definition to determine the requirements for expanding an existing polyVenn diagram.

Theorem 2.5. *Let* V *be a polyVenn on a* $2^r \times 2^c$ *grid* G. *Let* G *be expanded to* $X(G)$ *where every grid* $g_{ij} \in G$ *is replaced by a* $2^{r'} \times 2^{c'}$ *expansion,* G_{ij}, $0 \le i < 2^r$, $0 \le j < 2^c$. *Suppose, for every* G_{ij}, *there is an HSS* $M_{ij} = (M_{ij1}, M_{ij2}, \ldots, M_{ij(r'+c')})$ *on the unit squares of* G_{ij} *for which the following is true:*

For each $1 \le k \le r' + c'$,

$$E_k = \bigcup_{\substack{0 \le i < 2^r \\ 0 \le j < 2^c}} M_{ijk} \quad \text{covers a simply connected space on } X(G).$$

Then a $(r + r', c + c')$*-polyVenn exists that is a* $2^{r'} \times 2^{c'}$ *expansion of* V.

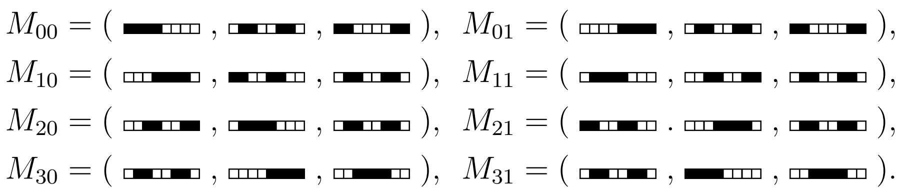
Proof. We represent the set of polyominoes in P in HSS form as described in Section 2.2. By Lemma 2.3, we know that $X(P) \cup E$ where $E = \{E_1, \ldots, E_{r'+c'}\}$ and $X(P)$ as defined in Equation 2.3 is an HSS on $X(G)$. Clearly, if each P_j is simply connected on G, then $X(P_j)$ is simply connected on $X(G)$. Since E is already defined as a simply connected set, $X(P) \cup E$ is a simply connected HSS on the $2^{r+r'} \times 2^{c+c'}$ grid and by Definition 2.4, is a $(r + r', c + c')$-polyVenn. $\square$

3. Two expansions

We have two expansion results for (r, c)-polyVenns, Theorem 3.1 and Theorem 3.2.

Theorem 3.1. *If there is a $(2, c)$-polyVenn, then there is a $(2, c + 3)$-polyVenn.*

Proof. Let V be a $(2, c)$-polyVenn. For each g_{ij} on the 4×2^c grid, let G_{ij} be a 1×8 expansion. Consider 8 sets M_{ij} where $0 \leq i < 4$ and $0 \leq j < 2$ and each M_{ij} is a set of 3 subsets of unit squares of a 1×8 grid. Instead of using the subsets of the labeled unit squares of a 1×8 grid, we use the following illustrations, where a unit square is in the subset if it is darkly colored. Let



For $2 \leq j < 2^c$, let $M_{ij} = M_{i0}$ when j is even and $M_{ij} = M_{i1}$ when j is odd. It is easy to check from the little grid drawings that each M_{ij} is an HSS.

Let E_k be the union of the k^{th} subset of each of the M_{ij} sets, where $0 \leq i < 4$, $0 \leq j < 2^c$ and $1 \leq k \leq 3$. In the M_{ij} sets shown above, we can see that each of the 1×8 rows, when stacked on top of each other in the order they are laid out, forms a connected piece. Also, we see that the set of grids on the right are horizontal reflections of the grids on the left.

When we arrange each M_{ij} subset into G_{ij}, we can visualize the connectedness of the arrangements. Figure 3 clearly demonstrates that each E_k is simply connected, for all values of $c \geq 1$. Therefore, by Theorem 2.5, any $(2, c)$-polyVenn can be expanded using the described construction. $\square$

Theorem 3.2. *If there is a (r, c)-polyVenn, then there is a $(r + 1, c + 1)$-polyVenn.*

Proof. Let V be an (r, c)-polyVenn on a $2^r \times 2^c$ grid G. For each g_{ij} on G, let G_{ij} be a 2×2 mini-grid on $X(G)$, the $2^{r+1} \times 2^{c+1}$ expansion of G. Consider a set $M_{ij} = \{M_{ij1}, M_{ij2}\}$ where each M_{ij1} and M_{ij2} represent a pair of orthogonal dominoes on the 2×2 grid. For ease of visualization, we continue to represent each subset of unit squares on a 2×2 grid as it would appear if we filled in each square. For instance the subset $\{(0, 0), (0, 1)\} \subset \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ would be represented as  . Consider the following definitions of M_{ij}, determined by 4 quadrants of $X(G)$,{upper left, upper right, lower left or lower right}.

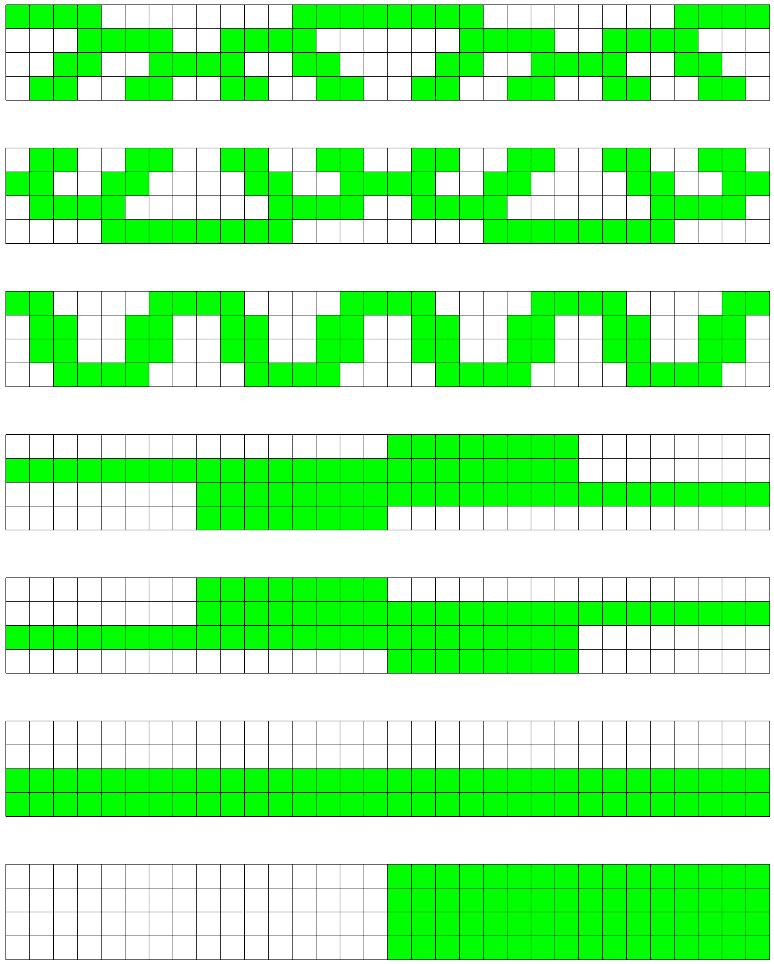


FIGURE 3. A 4×8 expansion of a $(2,2)$-polyVenn: the top 3 pieces are $\{E_1, E_2, E_3\}$; the bottom pieces are $\{X(P_1), X(P_2), X(P_3), X(P_4)\}$.

Upper left: $M_{ij} = (\ ,\ )$ or $(\ ,\ )$
Upper right: $M_{ij} = (\ ,\ )$ or $(\ ,\ )$
Lower left: $M_{ij} = (\ ,\ )$ or $(\ ,\ )$
Lower right: $M_{ij} = (\ ,\ )$ or $(\ ,\ )$

Clearly each M_{ij}, a set of one vertical and one horizontal domino, is an HSS on the 2×2 mini-grid. To meet the requirements for Theorem 2.5, we need an arrangement of the dominoes that is simply connected on the larger $2^{r+1} \times 2^{c+1}$ grid.

Figures 4 and 5 illustrate two such layouts. For every G_{ij}, there is vertical domino in E_1 or E_2 and a horizontal in the other. Note that the figures have the same layout in

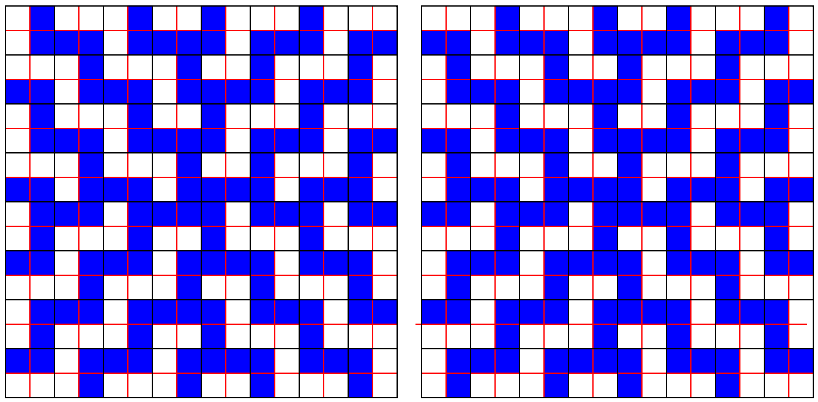


Figure 4. An $(r+1, c+1)$ expansion when $r = c = 3$.

the four center mini-grids, with the dominoes fitting together in a "spiral" formation. Both figures also divide the grid into 4 quadrants from the center.

In Figure 5, E_1 consists of the first subset of all the M_{ij}, determined by the quadrant of both i and j. E_2 is the union of all the second subsets of the left M_{ij} sets It is very clear that this layout is both connected and simple. Thus, by Thereom 2.5, the $(r+1, c+1)$-polyVenn exists.

Although only one example is sufficient, Figure 4 is a worthwhile construction to illustrate. It uses a combination of both M_{ij} options, alternating them between adjacent G_{ij} grids. Moving outward from the center, the connected wavy polygonal chains branch out only at the boundaries of the quadrants. Outside of the center and the quadrant boundary lines, each domino is connected to only two dominoes, forming the wavy polygonal chains that emanate outward from the center. On the boundary lines, three dominoes are connected, one branching into two.

If we represent the center four dominoes as a node in a graph, and every other domino as a node, with edges between dominoes that are connected, we have a tree graph with the center nodes as the root. Since a tree is connected and there are no cycles that would close the "holes", the diagram is simply connected. □

4. The base cases

In order for the inductive expansions to work we must start with some existing polyVenns. When $r = 0$, the largest value of c is 2, by Lemma 1.4. When $r = 1$, we make the following conjecture:

Conjecture 4.1. *A $(1, c)$-polyVenn does not exist when $c > 4$.*

The three polyVenns with $r = 0$ are shown in Figure 6. In Figures 8 and 9, we show a $(1, 3)$-polyVenn and a $(1, 4)$-polyVenn.

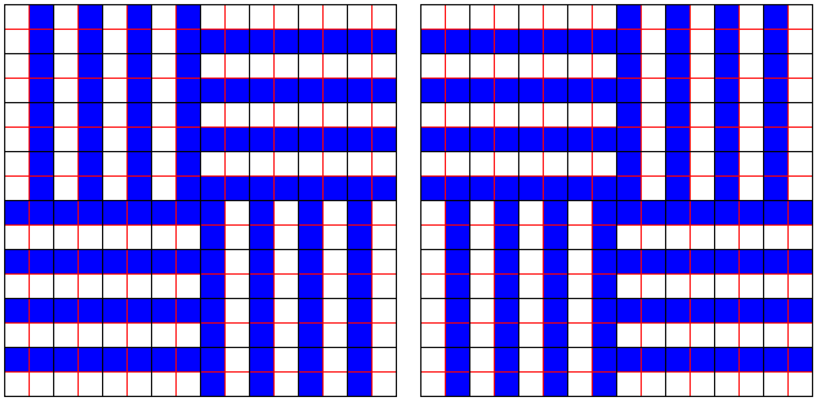


FIGURE 5. Another $(r+1, c+1)$ expansion when $r = c = 3$.

A	

B	**AB**	**A**	

FIGURE 6. All of the $(0, c)$-polyVenns.

	A
B	**AB**

C	**AC**	**B**	
BC	**ABC**	**AB**	**A**

FIGURE 7. The trivial $(1, c)$-polyVenns.

Now we have the base cases necessary to find (r, c)-polyVenns for all $r, c > 1$. See Table 1 for a grid that shows how all are found, either as base cases or from the expansion theorems.

5. Rectangles That Omit the Empty Set

In this section we discuss the problem of finding polyVenns that fit into $h \times w$ rectangles where $2^n - 1 = hw$, with no explicit square for the empty set. Obviously this is possible only when $M_n = 2^n - 1$ is *not* a Mersenne prime. The first 8 such numbers are $2^4 - 1 = 15 = 3 \cdot 5$, $2^6 - 1 = 63 = 3^2 \cdot 7$, $2^8 - 1 = 255 = 3 \cdot 5 \cdot 17$, $2^9 - 1 = 511 = 7 \cdot 73$, $2^{10} - 1 = 1023 = 3 \cdot 11 \cdot 31$, $2^{11} - 1 = 2047 = 23 \cdot 89$, $2^{12} - 1 = 4095 = 3^2 \cdot 5 \cdot 7 \cdot 13$. This problem was first mentioned in [2].

There is a simple construction of a polyVenn of dimension $(2^{n/2} - 1) \times (2^{n/2} + 1)$ when n is even. Consider the $(r+1, c+1)$ expansions described in Theorem 3.2 and shown in Figures 4 and 5. If we start from the base case in in Figure 6, we choose the empty set to be represented by g_{00}. In repeated expansions, the new g_{00} in each $X(G)$ also

	D	C D	C	A C	A C D	A D	A
B	B C	B C D	B D	A B D	A B C D	A B C	A B

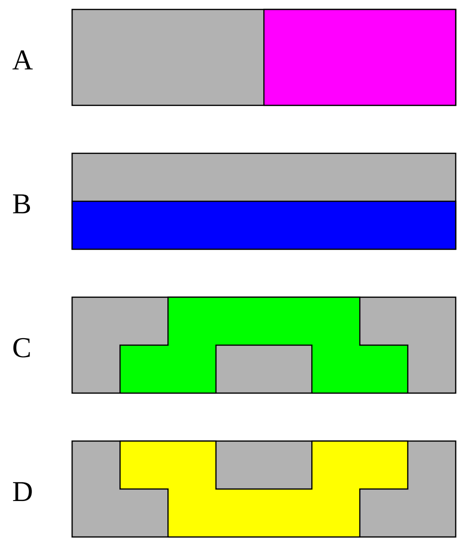


FIGURE 8. A $(1, 3)$-polyVenn with nice symmetry.

represents the empty set. We remove g_{00} from the grid, then strip off the first row, $g_{01} \ldots g_{0,2^c-1}$ from square polyVenn. We rotate it $90°$ and attach it to the last column as the new $g_{0,2^c} \ldots g_{2^r-1,2^c}$.

Figure 10 shows an example of the base case that converts the $(1, 1)$-polyVenn to the 1×3 grid. Figure 11 demonstrates this operation for $n = 8$, using the repeated 2×2 expansion illustrated in Figure 4. It is easy to see how the same technique applies to the expansion illustrated in Figure 5. The arrangements remain simply connected by this operation because of a rotational symmetry of all but 2 polyominoes in both expansions for all $r = c > 1$. The 2 non-symmetric polyominoes are the expansions of the base case. However, Figure 11 shows that the polyominoes remain connected under this operation.

The next open grid dimensions are

$$63 = 3 \times 21,\ 255 = 3 \times 85 = 5 \times 51, \text{ and } 511 = 7 \times 73.$$

These are not covered by the above construction.

It seems unlikely that there will be any general construction possible when n is odd and M_n is not prime.

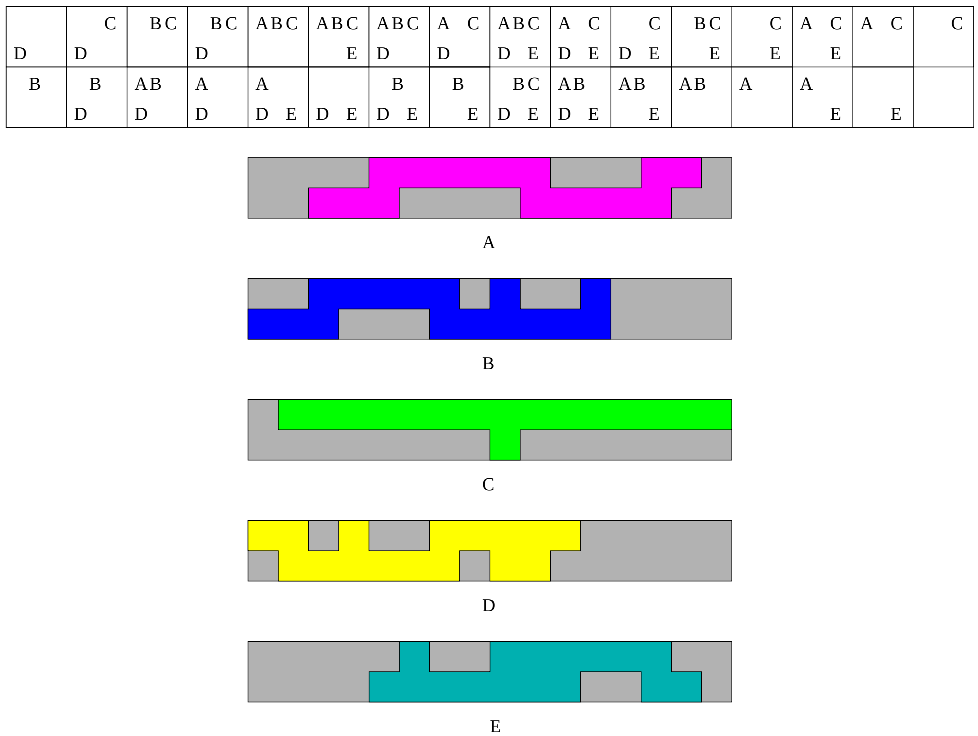


FIGURE 9. A (1, 4)-polyVenn.

	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
2^0	F6	F6	F6	X	X	X	X	X	X	X	X
2^1		F7	F7	F8	F9	?	?	?	?	?	?
2^2			B	B	B	A	A	A	A	A	A
2^3				B	B	B	B	B	B	B	B
2^4					B	B	B	B	B	B	B
2^5						B	B	B	B	B	B
2^6							B	B	B	B	B
2^7								B	B	B	B
2^8									B	B	B
2^9										B	B
2^{10}											B

TABLE 1. PolyVenn Grid Dimensions Key; Fn is the diagram shown in Figure n; X are configurations known to be impossible by Lemma 1.4; A and B are configurations implied by Theorems 3.1 and 3.2, respectively; The question marks are currently open problems.

Figure 10. Converting the base case $(1, 1)$-polyVenn to a 1×3 polyVenn that omits the empty set.

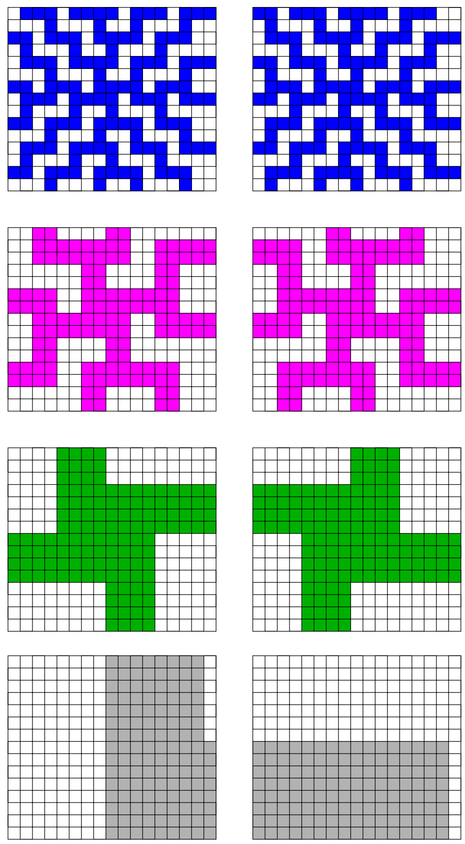


Figure 11. Omiting the empty set: a polyVenn on the $(2^{n/2} - 1) \times (2^{n/2} + 1)$ grid for $n = 8$.

6. Final Remarks and Open Problems

We are presently trying to prove the non-existence of a $(1, 5)$-polyVenn through both exhaustive computer search and mathematical techniques. If $\mathrm{num}(n)$ is the number of polyomino pieces with area 2^{n-1} and height no more than 2, up to vertical and horizontal reflection, then $\mathrm{num}(2) = 2$, $\mathrm{num}(3) = 6$, $\mathrm{num}(4) = 63$, $\mathrm{num}(5) = 8189$ and $\mathrm{num}(6) = 140{,}473{,}849$. If P_n denotes the set of distinct polyVenns (again up to vertical and horizontal reflection) on the $2 \times 2^{n-1}$ grid, then $|P_2| = 1$, $|P_3| = 34$, $|P_4| = 3034$. We are currently computing the number of polyVenns when $n = 5$; the computation has not finished, but the number is at least a million.

It is interesting to contemplate what happens in our construction for the case of a square, when $n \to \infty$. If we restrict each square so that it is 1×1 then note that our construction implies the following proposition, where $\{0,1\}^\infty$ is the set of all (one-way) infinite binary strings $b_1 b_2 \cdots$.

Proposition 6.1. *There is function $f : [0,1] \times [0,1] \to \{0,1\}^\infty$ with the property that each of the pre-images of the i-th projection to 1 is connected. I.e., for $i = 1, 2, \ldots$, each of the sets*

$$f^{-1}(\{b_1 b_2 b_3 \cdots \in \{0,1\}^\infty : b_i = 1\})$$

is connected.

7. Appendix: Proofs

7.1. **Proof of Theorem 2.2.** We start by showing that the following lemma demonstrates that we can substitute any S_i with its complement and this new collection of sets is also an n-HSS.

Lemma 7.1. *For an n-HSS $\{S_1, S_2, \ldots, S_n\}$, the following statement is true: For any subset $A \subseteq \{1, \ldots, n\}$ and any 2-partition (X, Y) of A,*

$$\left| \left(\bigcap_{i \in X} S_i \right) \cap \left(\bigcap_{i \in Y} \overline{S_i} \right) \right| = \left| \bigcap_{i \in A} S_i \right| = 2^{n-|A|}.$$

Proof. Our proof is by induction on increasing values of $|Y|$ and $|A|$. If $|Y| = 0$, then $X = A$ and $Y = \oslash$ and the statement is true by Definition 2.1. Assume that it is true for all $0 \le |Y| < |A|$. Let $e \in Y$ and

$$Z = \left(\bigcap_{i \in X} S_i \right) \cap \left(\bigcap_{i \in Y \setminus \{e\}} \overline{S_i} \right).$$

Note $|Z| = 2^{n-(|A|-1)}$ and $|Z \cap S_e| = 2^{n-|A|}$ by the induction hypothesis. Let

$$\begin{aligned} Z' &= Z \setminus (Z \cap S_e) \\ &= Z \cap (\overline{Z} \cup \overline{S_e}) \\ &= Z \cap \overline{Z} \cup Z \cap \overline{S_e} \\ &= Z \cap \overline{S_e}. \end{aligned}$$

The sets $Z \cap S_e$ and $Z \cap \overline{S_e}$ are disjoint, and $(Z \cap S_e) \cup (Z \cap \overline{S_e}) = Z$, so

$$\begin{aligned}
|Z \cap \overline{S_e}| &= \left| \left(\bigcap_{i \in X} S_i \right) \cap \left(\bigcap_{i \in Y \setminus \{e\}} \overline{S_i} \right) \cap \overline{S_e} \right| \\
&= \left| \left(\bigcap_{i \in X} S_i \right) \cap \left(\bigcap_{i \in Y} \overline{S_i} \right) \right| \\
&= |Z| - |Z \cap S_e| \\
&= 2^{n-(|A|-1)} - 2^{n-|A|} = 2^{n-|A|}
\end{aligned}$$

□

Copy of Theorem. *Let $\{S_1, S_2, \ldots, S_n\}$ be a collection of subsets of $\{1, 2, \ldots, 2^n\}$. For all $A \subseteq \{1, 2, \ldots, n\}$*

$$|\bigcap_{i \in A} S_i| = 2^{n-|A|}$$

if and only if for all subsets $B \subseteq \{1, 2, \ldots n\}$, there is a unique element $m \in \{1, 2, \ldots, 2^n\}$ such that

$$\{m\} = \left(\bigcap_{i \in B} S_i \right) \cap \left(\bigcap_{i \notin B} \overline{S}_i \right).$$

Proof. $(\Rightarrow)$ Let $B, C \subseteq \{1, 2, \ldots, n\}$ such that $B \neq C$. Let $j \in B$ and $j \notin C$. Let X and Y be subsets and

$$X = \left(\bigcap_{i \in B} S_i \right) \cap \left(\bigcap_{i \notin B} \overline{S_i} \right),$$

and

$$Y = \left(\bigcap_{i \in C} S_i \right) \cap \left(\bigcap_{i \notin C} \overline{S_i} \right).$$

By Lemma 7.1, $|X| = |Y| = 2^{n-n} = 1$. Let $X = \{m\}$ and $Y = \{n\}$. Since $m \in S_j$ and $n \in \overline{S_j}$, $m \neq n$. So B maps to a unique element m. Conversely, for any element $m \in \{1, 2, \ldots, 2^n\}$ there is a subset $\{i \mid m \in S_i\} \subseteq \{1, 2, \ldots, n\}$. Since $|\{1, 2, \ldots, 2^n\}| = |\{B \mid B \subseteq \{1, 2, \ldots, n\}\}|$, B is unique.

$(\Leftarrow)$ Let $S = \{S_1, S_2, \ldots, S_n\}$ be a set of subsets on $\{1, 2, \ldots, n\}$. Suppose that for all $B \subseteq \{1, 2, \ldots n\}$, there exists a unique value $m \in \{1, 2, \ldots, 2^n\}$ such that

$$\{m\} = \left(\bigcap_{i \in B} S_i \right) \cap \left(\bigcap_{i \notin B} \overline{S}_i \right).$$

Let $A \subseteq \{1, 2, \ldots, n\}$ and $B \subseteq A$. We use induction on k where $|A| = n - k$ and

$$|Z| = \left| \left(\bigcap_{i \in B} S_i \right) \cap \left(\bigcap_{i \in A \setminus B} \overline{S_i} \right) \right| = 2^k.$$

For the base case, $k = 0$. Then $|Z| = 2^0$, $|A| = |\{1, 2, \ldots, n\}| = n - 0$, and $Z = \{m\} \subseteq A$. Suppose the statement is true for all $k \geq 0$. Then let $|Z| = 2^k$, $|A| = n - k$ and $Z \subseteq A$. Choose a subset $B \subseteq A$ and let $j \in B$. Let

$$X = \left(\bigcap_{i \in B} S_i \right) \cap \left(\bigcap_{i \in A \setminus B} \overline{S_i} \right),$$

$$Y = \left(\bigcap_{i \in B \setminus \{j\}} S_i \right) \cap \left(\bigcap_{i \in A \setminus (B \setminus \{j\})} \overline{S_i} \right)$$

and $Z = X \cup Y$. Because X includes an intersection of S_j while Y includes an intersection of $\overline{S_j}$, X and Y are disjoint sets. Therefore $|Z| = |X \cup Y| = |X| + |Y| = 2 \times 2^k = 2^{k+1}$. Furthermore:

$$\begin{aligned} Z = & \left[\left(\bigcap_{i \in B \setminus \{j\}} S_i \right) \cap \left(\bigcap_{i \in A \setminus B} \overline{S_i} \right) \cap S_j \right] \cup \\ & \left[\left(\bigcap_{i \in B \setminus \{j\}} S_i \right) \cap \left(\bigcap_{i \in A \setminus B} \overline{S_i} \right) \cap \overline{S_j} \right] \\ = & \left(\bigcap_{i \in B \setminus \{j\}} S_i \right) \cap \left(\bigcap_{i \in A \setminus B} \overline{S_i} \right) \end{aligned}$$

Note $i \in A \setminus \{j\}$ and $|A \setminus \{j\}| = n - (k+1)$. Let $A \setminus \{j\}$ be partitioned into 2 subsets $C = B \setminus \{j\}$ and $D = A \setminus B$ and let

$$T = \{T_1, T_2, \ldots, T_n\}$$

be a set of subsets where

$$T_i = \begin{cases} \overline{S_i} & \text{if } i \in D \\ S_i & \text{if } i \in C \text{ or } i = j \text{ or } i \notin A. \end{cases}$$

Then

$$\begin{aligned}
\left|\bigcap_{i\in A\setminus\{j\}} T_i\right| &= \left|\left(\bigcap_{i\in C} T_i\right)\cap\left(\bigcap_{i\in D} T_i\right)\right| \\
&= \left|\left(\bigcap_{i\in B\setminus\{j\}} S_i\right)\cap\left(\bigcap_{i\in A\setminus B} \overline{S_i}\right)\right| \\
&= |Z| \\
&= 2^{k+1} \\
&= 2^{n-(n-(k+1))} \\
&= 2^{n-|A\setminus\{j\}|}.
\end{aligned}$$

So T is an HSS. By Lemma 7.1, S is an HSS. ☐

7.2. **Proof of Lemma 2.3.**

Copy of Lemma. *Let G be a set of $2^r \times 2^c$ unit squares on a grid, labeled $\{g_{i,j} \mid 0 \le i < 2^r, 0 \le j < 2^c\}$ Let $X(G)$ be an (r', c') expansion of G, where each mini-grid G_{ij} of dimension $2^{r'} \times 2^{c'}$ on $X(G)$ replaces g_{ij} on G. Let $P = \{P_1, P_2, \ldots, P_{r+c}\}$ be an HSS on G. Let $X(P)$ be a set of subsets, $\{X(P_1), X(P_2), \ldots, X(P_{r+c})\}$ on $X(G)$ where $X(P_j)$ has the following property:*

If $g_{ij} \in P_j$, then $G_{ij} \in X(P_j)$.

For every G_{ij} on $X(G)$, let $M_{ij} = \{M_{ij1}, M_{ij2}, \ldots, M_{ij(r'+c')}\}$ be an HSS on G_{ij}. Let E be a set of subsets on $X(G)$, $E = \{E_1, E_2, \ldots, E_{r'+c'}\}$ with the conditions that for all $k \in \{1, 2, \ldots, r' + c'\}$,

$$E_k = \bigcup_{\substack{0\le i<2^r\\ 0\le j<2^c}} M_{ijk}. \tag{7.1}$$

Then $X(P) \cup E$ is an HSS on $X(G)$.

Proof. Choose a unit square $g \in X(G)$. Then $g \in G_{uv}$ for some $0 \le u < 2^r$, $0 \le v < 2^c$. By Theorem 2.2, since M_{uv} is an HSS on G_{uv}, then there is a subset $B \subseteq \{1, 2, \ldots, r' + c'\}$ with the property that

$$\{g\} = \left(\bigcap_{k\in B} M_{uvk}\right)\bigcap\left(\bigcap_{k\notin B} \overline{M_{uvk}}\right). \tag{7.2}$$

Note that for any $i \ne p$ or $j \ne q$ that

$$M_{ij} \cap M_{pq} = \emptyset. \tag{7.3}$$

Because the M_{ij}s are distinct,

$$\overline{M_{ijk}} = M_{ij} \setminus M_{ijk},$$

and the complement of Equation (7.1) is defined as

$$\overline{E_k} = \bigcup_{\substack{0 \le i < 2^r \\ 0 \le j < 2^c}} \overline{M_{ijk}}. \tag{7.4}$$

Let $Y \subseteq X(G)$, where

$$\begin{aligned} Y &= \left(\bigcap_{k \in B} E_k \right) \bigcap \left(\bigcap_{k \notin B} \overline{E_k} \right) \\ &= \left[\bigcap_{k \in B} \left(\bigcup_{\substack{0 \le i < 2^r \\ 0 \le j < 2^c}} M_{ijk} \right) \right] \bigcap \left[\bigcap_{k \notin B} \left(\bigcup_{\substack{0 \le i < 2^r \\ 0 \le j < 2^c}} \overline{M_{ijk}} \right) \right] \quad \text{by Equation (7.1) and (7.4)} \\ &= \left[\bigcup_{\substack{0 \le i < 2^r \\ 0 \le j < 2^c}} \left(\bigcap_{k \in B} M_{ijk} \right) \right] \bigcap \left[\bigcup_{\substack{0 \le i < 2^r \\ 0 \le j < 2^c}} \left(\bigcap_{k \notin B} \overline{M_{ijk}} \right) \right] \quad \text{by Equation (7.3)} \\ &= \bigcup_{\substack{0 \le i < 2^r \\ 0 \le j < 2^c}} \left[\left(\bigcap_{k \in B} M_{ijk} \right) \bigcap \left(\bigcap_{k \notin B} \overline{M_{ijk}} \right) \right]. \end{aligned}$$

By Equation (7.2), $g \in Y$ when $i = u$ and $j = v$. So

$$Y \cap G_{uv} = \{g\}. \tag{7.5}$$

Also, since P is an HSS, by Theorem 2.2 there is a subset $A \subseteq \{1, 2, \ldots, r + c\}$ such that

$$\{g_{uv}\} = \left(\bigcap_{i \in A} P_i \right) \bigcap \left(\bigcap_{i \notin A} \overline{P_i} \right),$$

then

$$G_{uv} = \left(\bigcap_{i \in A} X(P_i) \right) \bigcap \left(\bigcap_{i \notin A} \overline{X(P_i)} \right),$$

by definition of $X(P_i)$.

Now, if we let $C = A \cup \{i + r + c \mid i \in B\}$ and $S = X(P) \cup E$, where $X(P_i) = S_i$ and $E_i = S_{i+r+c}$, then by Equation (7.5),

$$\begin{aligned}\{g\} &= Y \cap G_{uv}\\ &= \left(\bigcap_{k\in B} E_k\right) \bigcap \left(\bigcap_{k\notin B} \overline{E_k}\right) \bigcap \left(\bigcap_{i\in A} X(P_i)\right) \bigcap \left(\bigcap_{i\notin A} \overline{X(P_i)}\right)\\ &= \left(\bigcap_{j\in C} S_i\right) \bigcap \left(\bigcap_{j\notin C} \overline{S_i}\right).\end{aligned}$$

By Theorem 2.2, $S = X(P) \cup E$ is an HSS. $\square$

References

[1] Casimiro, Tenreiro, Monteiro, *Identification of pathogenesis-related ESTs in the crucifer downy mildew oomycete Hyaloperonospora parasitica by high-throughput differential display analysis of distinct penotypic interactions with Brassica oleracea*, Journal of Microbiological Methods, 66 (2006) 466–478.

[2] S. Chow and F. Ruskey, "Minimum Area Venn Diagrams Whose Curves are Polyominoes," *Mathematics Magazine*, 80 no 2 (April 2007) 91-103.

[3] B. Grünbaum, "Venn Diagrams and Independent Families of Sets," *Mathematics Magazine*, 48:12–23, 1975.

[4] B. Grünbaum, "Venn Diagrams I", *Geombinatorics*, Volume I, Issue 4 (1992) 5–12.

[5] M. Thompson, "Venn Polyominoes", *Math Recreations*, `http://home.flash.net/~markthom/html/venn_polyominos.html`.

[6] P. Winkler, "Venn Diagrams: Some Observations and an Open Problem," *Congressus Numerantium*, 45 (1984) 267–274.

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