Computer Science 428/528 Midterm, 2015 NAME: I.M. Solution

1. [1 mark] What famous computer scientist had the following dedication in their first book?

   > This series of books is affectionately dedicated to the Type 650 computer once installed at Case Institute of Technology, in remembrance of many pleasant evenings.

   ANSWER: DEK

2. [3 marks] How many *normal* n-input 2-output boolean functions are there?

   ANSWER: 2^{2^n-1}

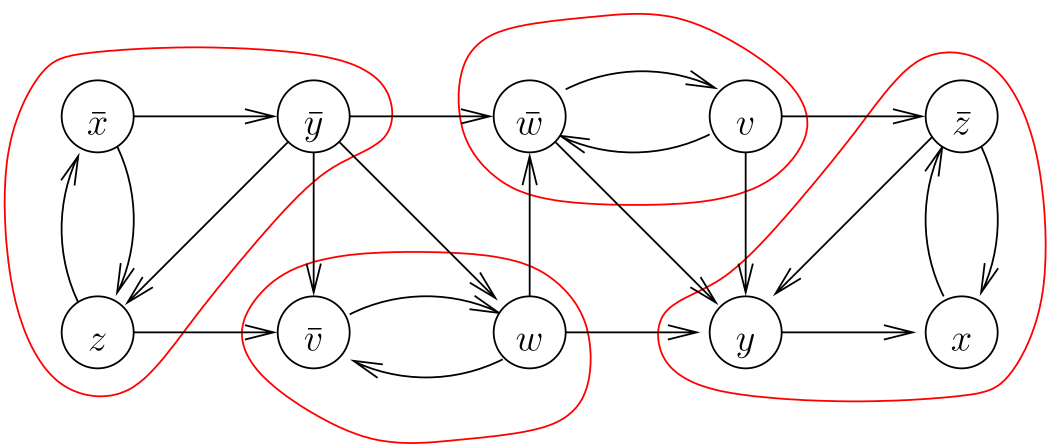
3. [3 marks] What is a good way (i.e., using just a few machine instructions) to determine whether an unsigned positive integer x is a power of 4?

   ANSWER: $x\& -x\&\mu_0 = x$ (recall $\mu_0 = \cdots 01010101$)

4. [2 marks] Write $\langle x, y, z\rangle$ in CNF and in DNF.

$$\langle x, y, z\rangle = \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{CNF}} = \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{DNF}}$$

5. [4 marks] The following graph arises in determining the solution to a Krom clause satisfiability problem. Circle the strongly connected components. Specify a satisfying assignment for v, w, x, y, z.



$v = \underline{1},\ w = \underline{0},\ x = \underline{1},\ y = \underline{1},\ z = \underline{0}.$

6. [3 marks] What is the *core* of the following set of Horn clauses?

$\{\ a \wedge d \Rightarrow b,\ a \wedge c \Rightarrow d,\ \Rightarrow a\ \}$ core $= \underline{\{\ a\ \}}$

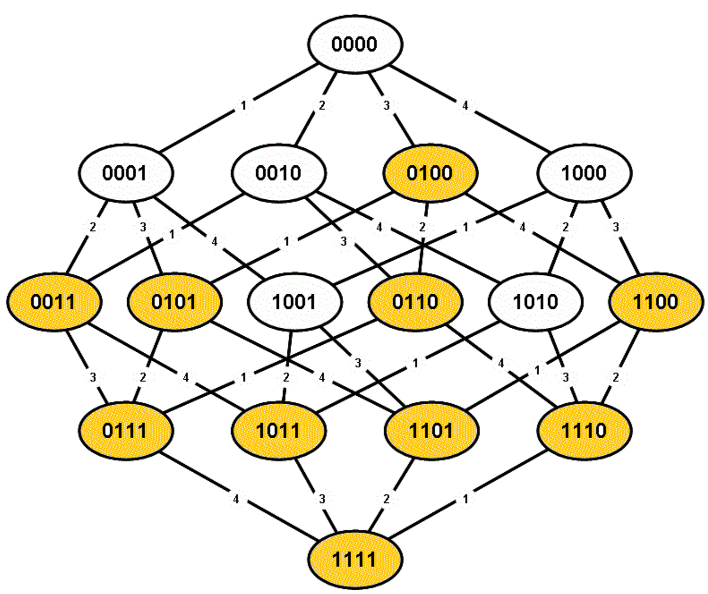
7. [5 marks] Is 01101001 the truth table of a symmetric function? Draw the (very simple) circuit diagram for this function.

   ANSWER: Yes, it is a symmetric function, namely $S_{1,3}(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_3$.

8. [5 marks] Formulate *graph coloring* as a SAT problem. Recall that in graph coloring each vertex of the graph gets a color and vertices that are incident on an edge must get different colors. Assume that there are d colors and that the vertices of the graph are $V = \{v_1, v_2, \ldots, v_n\}$ and the set of edges is E. Use variables x_{ij}, which signify that vertex v_i gets color j, for $1 \leq i \leq n$ and $1 \leq j \leq d$. Your task is to specify the clauses that make up the SAT expression that is true if and only if the graph $G = (V, E)$ is colorable with d colors. How many clauses are there in total as a function of n, d, and $m = |E|$?

   ANSWER:
   Vertex clauses: $x_{i1} \vee x_{i2} \vee \cdots \vee x_{id}$ for $i = 1, 2, \ldots, n$.
   Edge clauses: for each $(v_i, v_j) \in E$ add clauses $\bar{x}_{ik} \vee \bar{x}_{jk}$ for $k = 1, 2, \ldots, d$.

9. [6 marks] In the diagram below, a darkened vertex indicates that a boolean function f is 1 on the given input values, otherwise it is 0.



   - Is the function f monontone? Explain.

     ANSWER: Yes it is monotone; changing a 0 to a 1 at a darkened vertex always gives a darkened vertex.

   - Give the shortest DNF for $f(x_1, x_2, x_3, x_4)$. ANSWER: $x_2 \vee (x_3 \wedge x_4)$.

   - In general, what is the largest number clauses that are ever required in the shortest DNF expression for an n-variable monotone boolean function $f(x_1, x_2, \ldots, x_n)$? (E.g., looking at the diagram above, what is it for $n = 4$?)

     ANSWER: The largest number is the middle binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$. Need to know Sperner's lemma to have a true proof of this.