Determining an Aesthetic Inscribed Curve

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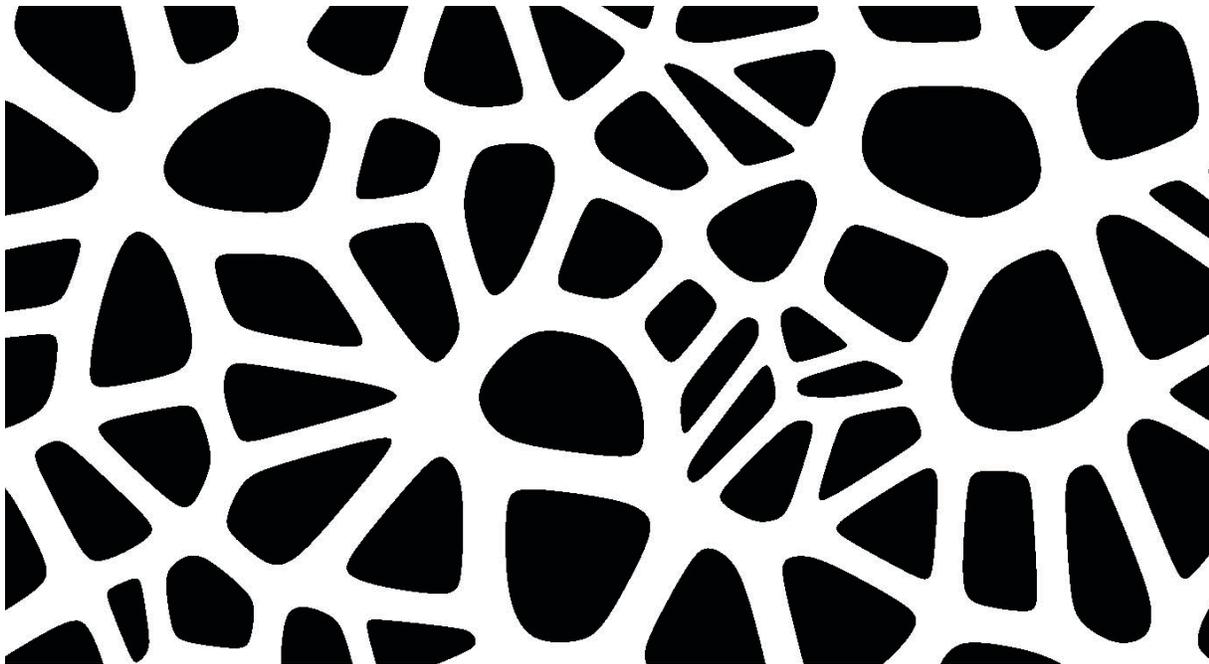[4]Carleton University, Canada.

Figure 1: *The image that garnered the most votes (63) as the most aesthetic.*

Abstract
In this work we propose both implicit and parametric curves to represent aesthetic curves inscribed by Voronoi cells in $\mathbb{R}^2$. A user survey was conducted to determine, which class of curves are generally accepted as the more aesthetic. We present the curves, the survey results, and the implications for future work on simulating sponge like volumes.
Leave one blank line after the abstract, then add the subject categories according to the ACM Classification Index (see http://www.acm.org/class/1998/).

Categories and Subject Descriptors (according to ACM CCS):   I.3.3 [Computer Graphics]: Picture/Image Generation—Line and curve generation

1. Introduction

Given a set of seed-points in the plane, the region around each seed can be separated from its neighbour by drawing a line equally distant from both points, giving rise to the familiar Voronoi diagram. The lines form *cells* around each seed-point, known as Voronoi regions. These regions in both 2*D* and 3*D* can be used to describe many natural formations, such as soap bubbles, sponges, crystals or bone cells, [OBSK00, Kle87], or the distribution of galaxies, [VJM02]. Architects, such as Toyo Ito, Buckminster Fuller or Frei Otto, took the geometry of the Voronoi cell, as constructive inspiration for their architecture, [AL06]. In some of these works, areas have been defined, bounded by a closed convex curves, which are in turn inscribed inside a Voronoi cell, see Figure 2. Craig Kaplan, [Kap99], uses the Voronoi digram as an artistic tool, and applies it to construct interesting tilings of the plane. In his paper he gives several examples of other artistic applications.

The current work is a stepping stone towards building aesthetic convex curved volumes, inscribed in 3*D* Voronoi cells. Since we had a ready access to a 3D volume modelling system based on the *BlobTree* , [WGG99], and we wanted to explore the empty, curved spaces, subtracted from the Voronoi cells, we decided to use this skeletal-implicit system. The approach was to define a suitable, procedural implicit primitive to represent the curved volumes. The main challenge was to design a primitive that would generate a suitable aesthetic volume. Our approach was first to look at the 2*D* problem, and use the results of this study to guide the definition of the 3*D* function.

Although parametric B-splines have been the norm, we wondered whether inscribed implicit curves would yield more appealing results. Architectural and art applications frequently use inscribed parametric B-spline curves, as they are offered by the available software packages, as can be seen on various websites: [Bil,Inc,CTEG]. We experimented with a number of different curve types, both parametrically and implicitly defined. We narrowed this down to three curve types, each with a number of parameters that govern their precise shape within a Voronoi cell. The next step was to compare the images generated from these different curves, in an experiment to discover the most aesthetically pleasing approach.

The rest of the paper is organized as follows: some previous work about aesthetic curve design, pattern creation, and evaluation of aesthetics is given in section 2, section 3 provides details of the curves we used and why, section 4 provides details of the user study, section 5 gives the results, and finally the conclusions and future work gives information on what we learned about aesthetic curves and the next step towards the larger project.
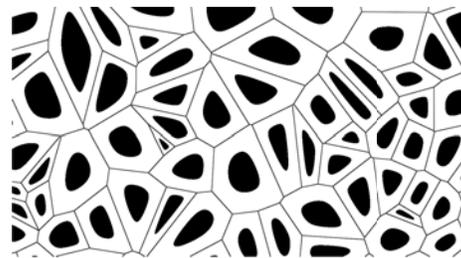

Figure 2: *Closed curves inscribed in Voronoi cells.*

2. Previous Work

Aesthetic appeal of planar curves has been long studied in the field of industrial design, where it is usually called "fairness" [FRSW87]. Fair curves are those where the curvature changes monotonically or piecewise monotonically [PN77, FRSW87, KNS03]. The monotone pieces may have linear curvature changes, in which case the curve is a piecewise clothoid [FRSW87, MS08]. Log-linear distributions of curvature have also been postulated [Har97, HYM99, KNS03, Miu06] as both reflecting the beauty of objects both natural and man-made, including bird's eggs and samurai swords [Har97], and of curves drawn by designers in practice [KNS03]. It is possible to adjust parametric curves to increase fairness [FRSW87, MS08], though we have less direct control over the shape of implicit curves.

In our application, we are interested in patterns formed by closed curves rather than individual curves. Wong et al. [WZS98] articulate the important principles of *order*, *repetition*, *balance*, and *conformation to constraints* as influencing the appeal of a pattern. The order and repetition of our patterns are controlled by the Voronoi cells, but the balance between foreground and background can be adjusted by changing the curves, and the curves also adhere to greater or lesser extent to the boundaries of the Voronoi partition. However, Wong et al. were establishing motivation for their own pattern generation system rather than attempting to give a general system for evaluating patterns, and it is difficult to use these principles to make subtle distinctions between our patterns.

In evaluating the aesthetic quality of photographs, user studies – i.e., asking people directly what they think of the images – has been the norm. Savakis et al. [SEL00] conducted one of the first such studies, attempting to determine the factors at play in decisions to include or exclude photographs from photo albums. More recently, Datta et al. [DJLW06] and Ke et al. [KTJ06] attempted to create automated photograph rating algorithms, treating user ratings as ground truth. Such algorithms have had limited success to date; certainly, they are not yet capable of judging aesthetics with the reliability and granularity to replace human judgement. In our work, we also relied on users' subjective evaluation to distinguish between the patterns we created.

3. The Curves

Our goal is to inscribe into each cell of a Voronoi diagram a closed curve in an "aesthetically pleasing" manner. It is inherently impossible to turn this condition of "aesthetically pleasing" into exact mathematical conditions on those curves. The previously mentioned curvature conditions are not particularly sensible in the presence of inscription constraints. However, it seems sensible to at least require such a curve to be smooth and to bound a convex region. This region must be contained in the Voronoi cell and should "approximate it from the inside" for some suitable notion of approximation.

In our approach we will consider each Voronoi cell individually and independently of the others. As a matter of fact we will even ignore the fact that we are dealing with a Voronoi cell but consider general convex polygons. So let P be a convex polygon with vertex set $p_0, p_1, \ldots, p_{k-1}$ in counterclockwise order around P and line ℓ_i supporting the edge $p_i p_{i+1}$ (with index arithmetic modulo k). Let n_i be the inward unit normal vector of ℓ_i and for $x \in \mathbb{R}^2$ define $d_i(x) = \langle x - p_i, n_i \rangle$ to be the signed distance of point x to line ℓ_i. We have $\ell_i = \{x \in \mathbb{R}^2 \mid d_i(x) = 0\}$ and P consists of exactly those points x for which $d_i(x) \geq 0$ for all i.

We will consider two different approaches for inscribing "nice" curves into P. One produces paratmetric curves, the other implicit curves.

3.1. Implicit Curve Approach

The initial idea is simple: Produce a function $F : \mathbb{R}^2 \to \mathbb{R}$ that is 0 on the boundary of P and positive in the interior of P, and as inscribed curve use some level set of F, i.e. a set $C_a = \{x \in P \mid F(x) = a\}$ for some appropriately chosen positive a. With an appropriate choice of F, in particular if it is smooth and convex over P, the produced curve will be smooth and convex also.

A natural choice for F appears to be

$$F(x) = \prod_{0 \leq i < k} d_i(x)$$

which obviously fulfills the conditions that it is 0 on the boundary of P and positive in P's interior. Since F is smooth over the interior of P we get smooth level set curves C_a for positive a. But are the curves convex? It turns out they are, although it need not be the case that F (or its negative) is convex over P's interior †. A simple trick allows to easily prove

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† To see possible non-convexity of F consider its restriction to a horizontal line $x_2 = c$ in the (x_1, x_2)-plane. We now have a polynomial of degree k in x_1 and in the graph of such a polynomial the arc connecting two consecutive roots can contain an inflection point and is then neither convex nor concave.

convexity of the level set curves: Consider $f(x) = \log F(x)$. Then $C_a = \{x \in P \mid f(x) = \log a\}$ and

$$f(x) = \log \prod_{0 \leq i < k} d_i(x) = \sum_{0 \leq i < k} \log d_i(x)$$

is convex since $-\log d_i(x)$ is convex over the interior of P for each i and the sum of convex functions is convex. Therefore each C_a is the level set of a convex function and hence it is convex. Considering f instead of F in the computation of C_a has an additional advantage in that f is only defined over the interior of P whereas F is defined on the entire plane. Thus $C_a = \{x \in \mathbb{R}^2 \mid f(x) = \log a\}$, but $\{x \in \mathbb{R}^2 \mid F(x) = a\}$ will in general also contain curves in other cells of the arrangement formed by the lines ℓ_i. As an aside, the function $f(x)$ is also known as *logarithmic barrier function* and plays an important role in linear optimization. Figure 3 shows an example 6-sided polygon with level curves of F (or equivalently f) for three different values of a.
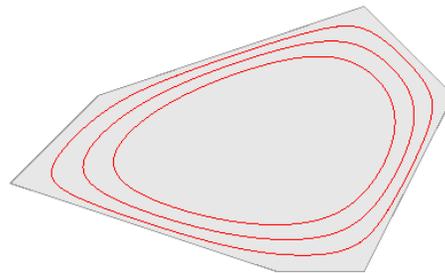

Figure 3: *Three example curves.*

It is now clear that for each positive a the curve C_a is smooth and convex and is contained in P, unless a is too large and C_a is actually empty. Moreover, as a approaches 0 the curve C_a approximates the (non-smooth) boundary of P arbitrarily closely. How do we choose the parameter a so as to get a particularly pleasing curve? An easy approach is to choose inside polygon P a point z through which we wish the curve C_a to pass. Obviously we then have $a = F(z)$. In our experiments we automatically produced z by considering convex combinations $\lambda p_1 + (1 - \lambda)c$ between the corner p_1 and a putative center of c of the polygon, namely the average of the corners. We ended up using $\lambda = 0.7$.

3.1.1. Generalizations

From a mathematical-aesthetic point of view our approach has the shortcoming that the approximating curves do not change continuously as the shape of P changes. For instance, if you replace a corner of a polygon P by a tiny edge the shape of P hardly changes however the approximating curve may change drastically, since the new edge contributes to f as strongly as all other edges. Figure 4 illustrates this effect by replacing the leftmost vertex of the polygon of Figure 3 by a short edges.
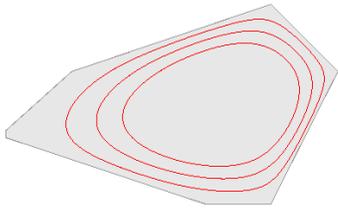
Figure 4: *Replacing the leftmost vertex by a tiny edge.*

A natural way to prevent this effect is to use edge lengths as weights in the definition of f, resulting in

$$f(x) = \sum_{0 \le i < k} \ell_i \log d_i(x)$$

where ℓ_i is the length of the i-th edge. Figures 5 and 6 illustrate how such a weighting prevents undue influences of short edges.
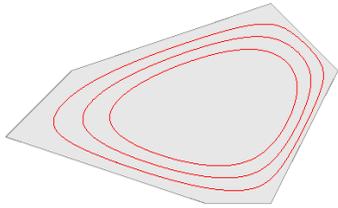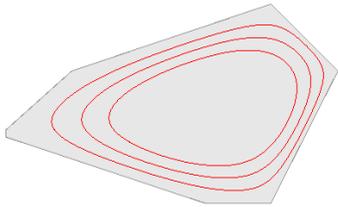

Figure 5: *Taking into account edge lengths.*

Figure 6: *Replacing the leftmost vertex by a tiny edge does not affect the curves very much.*

Our approach allows other variations also. The goal is to take level sets of a function that is convex over the interior of P and is unbounded on the boundary of P. A generic way to obtain such a function f is to consider

$$f(x) = \sum_{0 \le i < k} \ell_i h(d_i(x))$$

where h is a real valued function over the reals that is convex over the positive reals and is unbounded at 0. Examples of such functions are $h(u) = -\log u$, or $h(u) = 1/u^b$ with $b > 0$, or $h(u) = 1/(e^u - 1)$. For the purpose of this paper we just stick with the simple logarithmic case. Figures 7 through 9 show the resulting curves for various choices for $h(u)$. Note

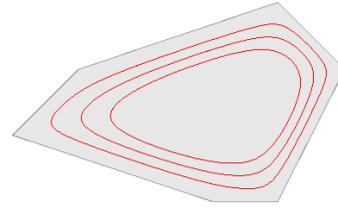how in the last figure, where $h(u) = 1/u^{10}$ the curves appear to be smooth approximations of offset curves.
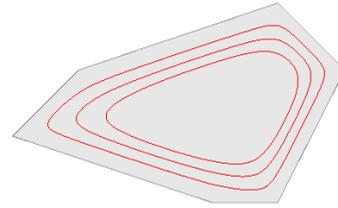

Figure 7: $h(u) = 1/u.$
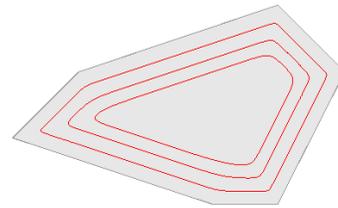

Figure 8: $h(u) = 1/u^2.$

Figure 9: $h(u) = 1/u^{10}.$

Let us consider again the general weighted definition

$$f(x) = \sum_{0 \le i < k} \ell_i h(d_i(x))$$

An equivalent definition of f is the following. Let $p(t)$ be the arc-length normalized parametric representation of the boundary curve of the polygon P, with $0 \le t < L$, where L is the length of the boundary. Furthermore, let $n(t)$ be the unit inward normal vector at point $p(t)$, except for the finitely many corners of P where the normal is not defined. Then

$$f(x) = \int_0^L h(\langle x - p(t), n(t) \rangle) dt \qquad (1)$$

The argument of h is the distance of point x to the tangent line of P in boundary point p. Note that this definition does not rely on P being a convex polygon. P could be *any bounded convex region,* in particular a region with a boundary containing corners and curved edges, see Figure 10 for a simple example the region is bounded by a parabolic arc and two segments. The function f will always be smooth and convex and can be used to define smooth convex inner approximations for any such convex region. Also note that

f is invariant under rigid motion. It also respects scaling, provided *h* satisfies the scaling condition that for every constant $a > 0$ there are constants $\alpha > 0$ and *beta* so that for all $u > 0$ we have $h(au) = \alpha h(u) + \beta$. Note that this holds for $h(u) = -\log u$ and $h(u) = 1/u^b$.

Finally, note that all our definitions generalize naturally to $\mathbb{R}^3$ and also to higher dimensions.
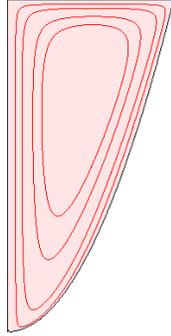

Figure 10: *Inscribed curves in a non-polygonal convex region.*

3.2. Parametric Curve Approach

In the parametric curve approach, the points defining the convex polygon can be used to build a smooth function of the form $c : \mathbb{R} \to \mathbb{R}^2$. We use a degree *n* B-spline basis and use the polygon points as control points, thus we can write the curve as

$$c(t) = \sum_{i=0}^{k+n} p_{(i \bmod k+1)} \, N_i^n(t), \quad t \in [0, k+n]. \quad (2)$$

We use a uniform knot vector, and we repeat the first *n* control points at the end of the sum in order to create a closed curve. The DeBoor Cox algorithm can be used to compute the basis functions $N_i^n(t)$ with a simple recursive formula.

Spline curves have a number of nice properties. The curve will be bounded by the convex hull of the control polygon. Furthermore, given a convex control polygon, the curve itself will be convex. The curves are piecewise polynomials, and C^{n-1} at the points between segments. Another property is that control points only influence the shape of the curve locally. This is a useful property when editing shapes.

Our control polygons are always convex, and it is interesting to note that higher order B-spline curves are smoother (despite being represented by polynomials of higher degree), and likewise these inscribe curves will lie further from the control polygon than lower order curves as shown in Figure 11. Unlike the barrier curves in the previous section, this does not give us a continuous parameter for selecting the amount of area between the inscribe curve and the polygon,
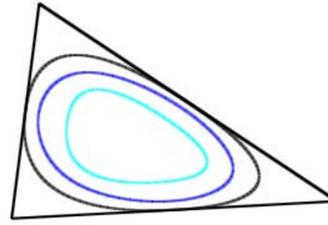

Figure 11: *An example of spline curves of degrees 2, 3, and 4.*
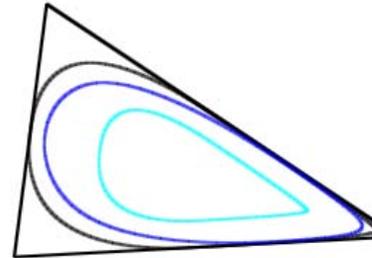

Figure 12: *An example of spline curves of degrees 2, 3, and 4 where the control polygon involves a small edge.*

thus we add a scale parameter. That is, we scale the whole curve about a centre of the polygon defined by the average of the control points to increase the space between the inscribing curve and the boundary. Figure 13 shows curves scaled at a value of 0.7, which we selected as producing a reasonably similar curve boundary separation as the barrier curves with α set to 0.7.

One disadvantage of B-splines is that adding a short edge inserts a new point and will have a noticable influence on the shape of the curve. A short edge of length ε effectively drops the continuity of the curve by one, and thus, a sequence of *n* short edges of length ε will act much like set of superimposed control points, causing the piecewise curve to interpolate the point with discontinuous derivative (see Figure 12).

While the initial definition of implicit curves in the previous subsection can be modified to take into account short edge lengths, a similar modification for spline curves is not straightforward. The difficulty is that the splines are parametric curves defined by control points, while the shape of the implicit curves is directly a function of the edges. The use of non-uniform knot vectors and weights to define rational splines, i.e., NURBS, are well known methods for providing additional control over the shape of B-spline curves [PT97]. But this does not provide an easy solution to the short edge problem because weights and knot values are assigned at the control points. Assigning small weights to two control points
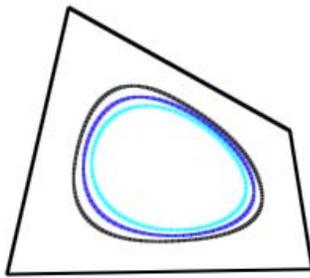
Figure 13: *An example of spline curves of degrees 2, 3, and 4 where the inscribing curves have been scaled by 0.7 about a centre of the control polygon computed as the average of the control points.*

adjacent to a small edge reduces their influence, but not in a manner equivalent to removing one of the control points. Likewise, knot values can be adjusted for edge lengths to change the region of influence of a control point, but again, not in a manner that would let us smoothly transition from a small edge to a single control point.
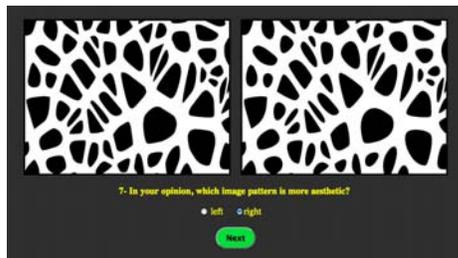

Figure 14: *Comparison of two typical images presented to the user in survey one.*

4. The User Study

Because humans are an integral part of defining what is aesthetic, we decided to employ a user study to find the most aesthetic curves appropriate to the overall objectives of this research, We started by generating images based on the same set of Voronoi diagrams, using barrier curves and cubic B-spline curves as described above.

The objective of the user study was to see if there was a clear preference for either parametric or implicit curve definitions. Two separate studies were undertaken. In the first we chose 30 pairs of images and asked the user to choose the most aesthetically pleasing (see Figure 14). The limit of 30 pairs, was the most we could expect without the user becoming too fatigued to make an informed choice or quit the study early [Zha10]. The second study asked the user to score each image according to a sliding scale of aesthetic value, using a
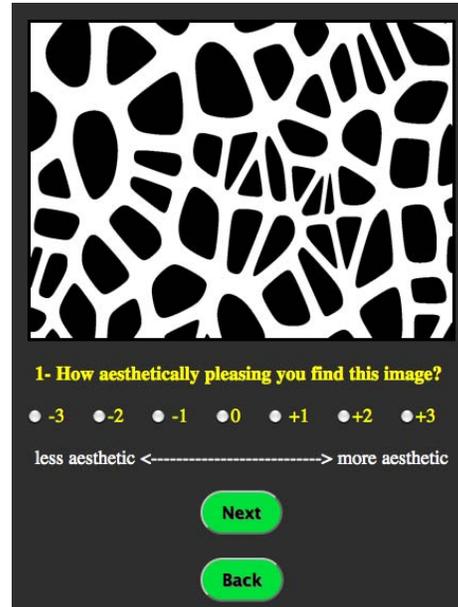

Figure 15: *A typical image presented to the user in survey two using the Likert.*

Likert scale [CP07] in the range [3 : +3], (see Figure 15). In both surveys the order of images was randomized, and the studies made available on the internet. Over one hundred responses were garnered for each survey.

In the first study we controlled against random choices by repeating some A/B pairs as B/A. If the choices were different we would eliminate that user from our statistics, but in practice this condition did not occur. In the second study no such test was available, but since the results were fairly consistent over 100 responses, and also because we simply wanted to determine whether the implicit curve or the parametric curve was more aesthetic, that effect would not alter the results by more than the a few percent. As can be seen in section 5, the user choices fall very clearly on the side of the implicit curves in both surveys. Since the result is so clearcut, small error effects may be disregarded. We also decided against doing a more detailed statistical analysis that would have been warranted if the data had shown only a marginal preference. Since it so clearly answered our question there was no need for more subtle analysis.

All curves were compared at similar scales and a range of iso-values for the implicit curves were compared against parametric curves, where the distance to the bounds of the Voronoi cells produced a similar set of curves.

5. Results

Figure 16 shows a summary of the results of the first survey. The *B* curves are the set of implicit curves, the *S*3 and *S*2
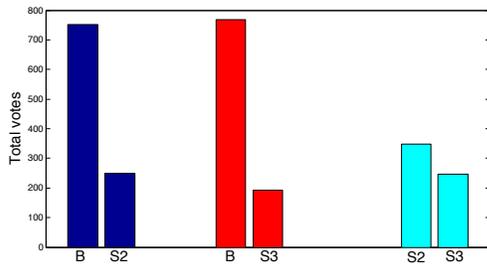
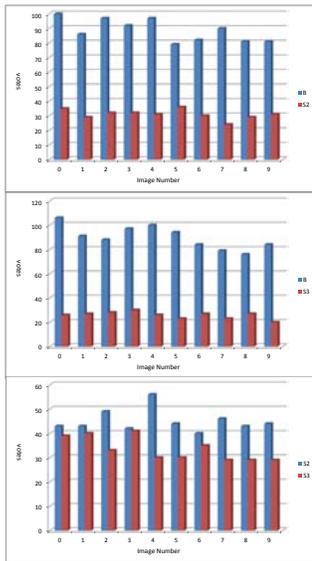**Figure 16:** *A summary of the three comparisons done in the first survey.*

Figure 17: *Details of the comparisons between implicit, quadratic and cubic curves.*

Figure 18: *The number of votes for each image from survey two, grouped as implicit (B-Blue), quadratic B-spline (red) and cubic B-spline (green).*

Figure 19: *The image that garnered the least votes (49).*

sen most often in both surveys is shown in Figure 1. The least most chosen images are shown in Figure 19.

There were 116 respondents for survey one, total number of questions answered was 3,318. The following results were obtained:

The number of preferences for the implicit curves was 1,887 representing 56.87% of the votes.
The number of preferences for the quadratic curves was 806 representing 24.29% of the votes.
The number of preferences for the cubic curves was 625 representing 18.84% of the votes.

There were 116 respondents for survey two, the following results were obtained:

The score for the implicit curves was +376.
The score for the quadratic curves was -128.
The score for the cubic curves was -227.

This clear preference for the implicit curve construction over the B-spline construction begs for some explanation. Here are some possible reasons. (1) Our B-spline curves are only C^2-continuous, whereas the the implicit curves are smooth, i.e. C -continuous. (2) Using the average of the

curves, the cubic and quadratic B-splines respectively. It can be seen in Figure 16 that the implicit curves were selected about three times as often as the quadratic curves in direct comparison, the implicit curves were selected almost four times as often as the cubic curves and that there was only a slight preference of the quadratic over the cubic curves when compared directly.

Figure 17 shows the details of the three comparisons.

The results of survey two are shown in Figure 18. Again a very clear preference for the implicit curves over the B-spline curves is shown by the survey results. There were 116 respondents for survey one and 102 for survey two. The results of both surveys showed an overwhelming preference for the implicit curves over the parametric. The image cho-
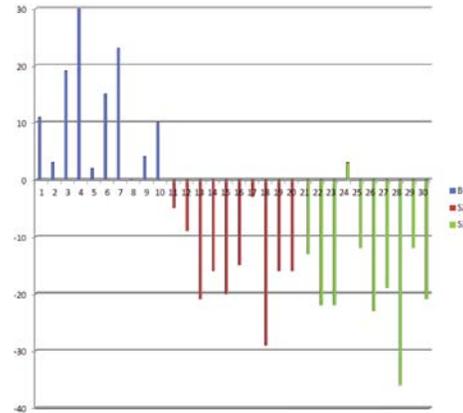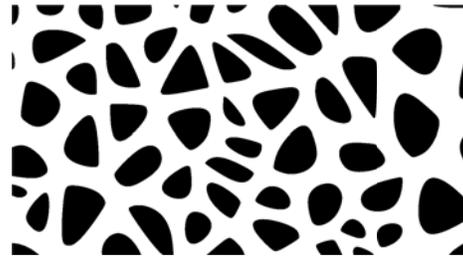
corners of the polygon as the centre for scaling the B-spline curve may be an unfortunate, geometrically not particularly meaningful choice. (3) You can argue that our implicit method is mathematically the more appropriate approach since it provides a continuous mapping from *convex shapes* to inscribed curves, whereas the B-spline construction provides a mapping from *sequences of points* to curves. It may be the case that this higher appropriateness leads to "better" curves and thus the aesthetics of mathematics are reflected in the aesthetics of human perception.

A few observations could also be made of the most popular image (Figure 1) versus the least popular (Figure 19). In Figure 1 the neighbouring cell borders are close to parallel. It gives the sense of a more unified image rather than a lot of unrelated objects that happen to be nearby, and also the channel widths seem to be more uniform than the more irregular channel widths of Figure 19. The sharp corners in Figure 19 could be considered to be a defect, and the irregularity of corner sharpness perhaps gives a bad impression. In contrast in Figure 1 the corners are all the same, whereas you can't predict the sharpness of the next corner in Figure 19, so it is perhaps seen as ugly.

6. Conclusions and Future Work

In order to design an implicit primitive suitable to build volumes enclosed by Voronoi cells in $\mathbb{R}^3$, it was proposed to first test curves in $\mathbb{R}^2$ to determine if there was an aesthetic preference for implicit over quadratic or cubic B-spline curves as the basis for designing the implicit primitives. Although there were some sources of possible error as discussed, the results showed an overwhelming preference for the implicit curves over either of the other two.

The next step in the project is to build an implicit primitive in $\mathbb{R}^3$, corresponding to the parameters for the curve chosen most often as the most aesthetic in both surveys.

7. Acknowledgements

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