Polyominoes, Gray Codes, and Venn Diagrams

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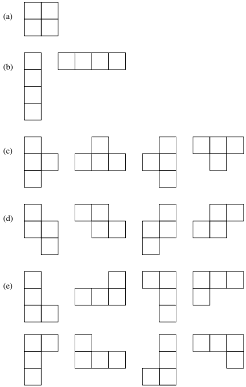
University of Victoria, CANADA

Northwest Theory Day, April 2005

Definition - polyomino

- Made from unit squares joined along edges.
- No holes allowed.
- Must be connected.
- Translations allowed but not flips or rotations.



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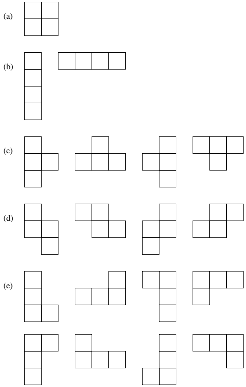

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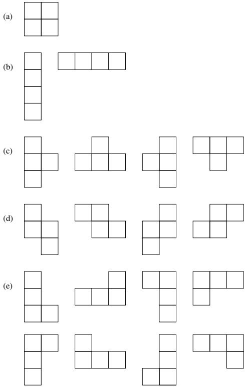

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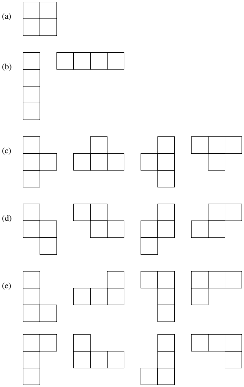
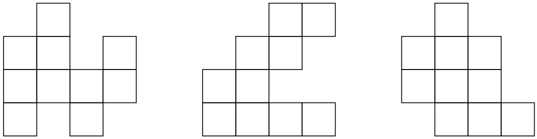


Definition - convexity conditions

- Column convex if intersections with vertical lines are connected.
- Satisfy the recurrence: $a_n = 5a_{n-1} - 7a_{n-2} + 4a_{n-3}$ ([Polya],nice proof:[Hickerson]).
- Convex if both row and column convex.
- *Gray codes: column convex only.*
- Further restriction: number of cells in each column is fixed.



(a) column–convex not row–convex

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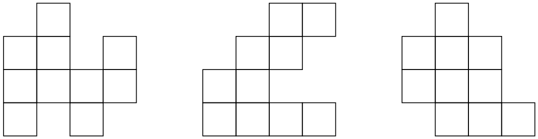

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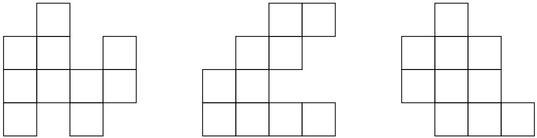

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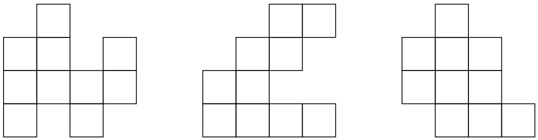

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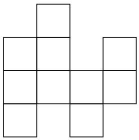

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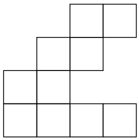
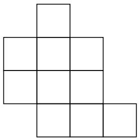


(b) row–convex not column–convex



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- Not unique $[1, 3, 1, 3] \rightarrow \langle 3, 3, 3 \rangle$ and $[2, 2, 2, 2] \rightarrow \langle 3, 3, 3 \rangle$.
- No polyomino for $\langle 1, 2, 1 \rangle$.
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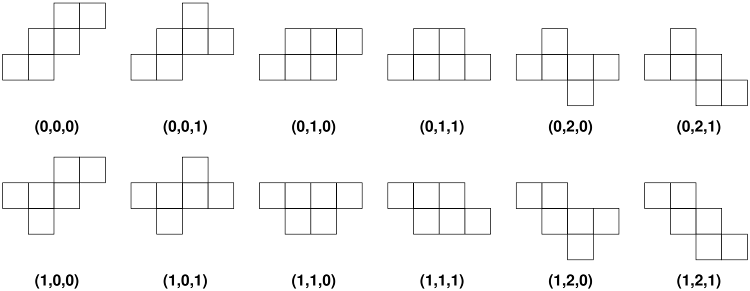

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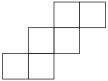

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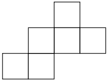

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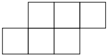

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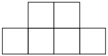

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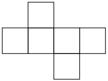

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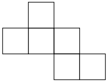

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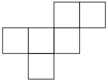

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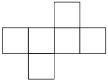

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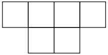

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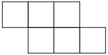

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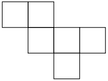

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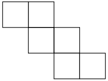

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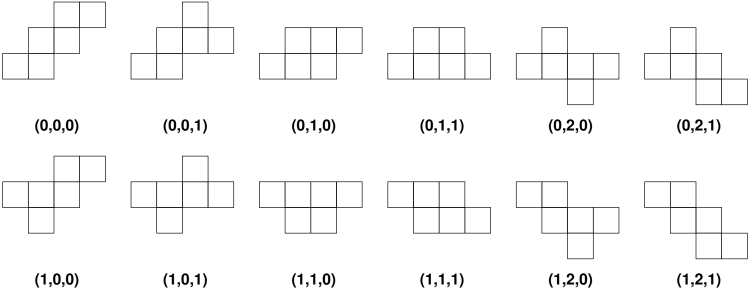

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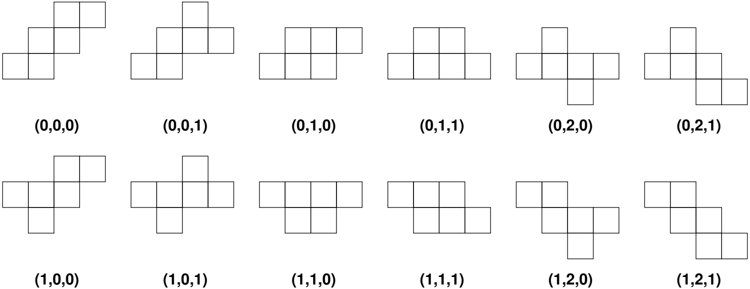

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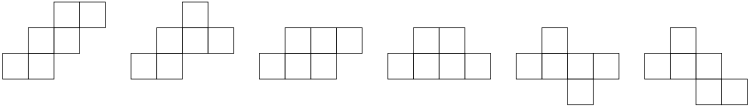

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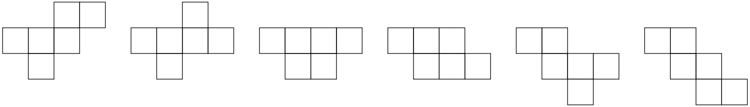

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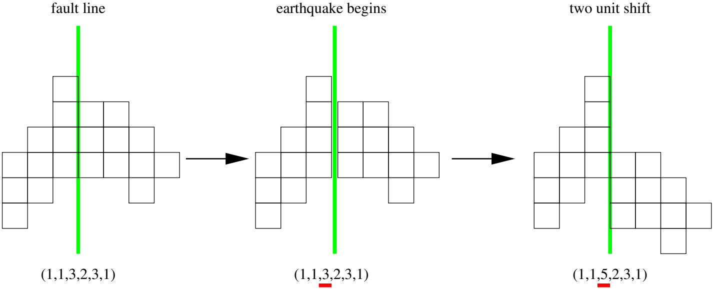
- The change $p_i := p_i \pm 1$ corresponds to a small earthquake.



- Can generate these using Gray codes for mixed-radix numbers (H-path in $k-1$ dimensional grid graph).
- A more interesting move is a single cell move within a column.
- Then $p_i := p_i \pm 1$ and $p_{i+1} := p_{i+1} \mp 1$ (except at extremities).

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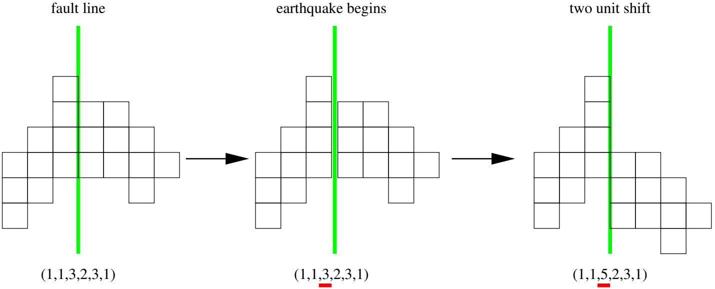

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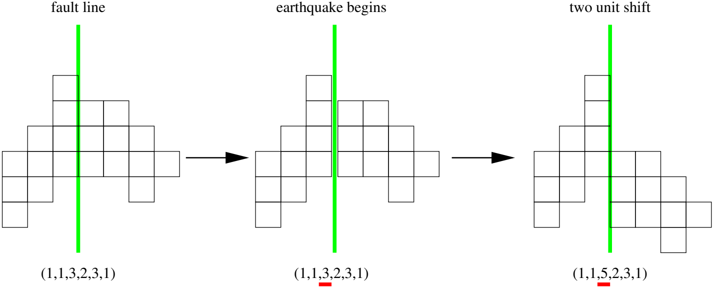

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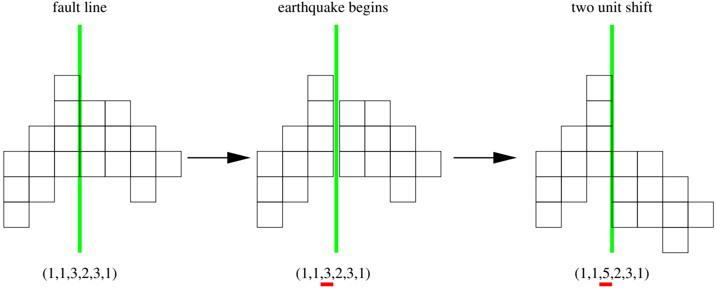

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- Define two operations (and their inverses) on **A**-sequences:
  $\tau_i : p_i := p_i + 1$ (only at the endpoints)
  $\sigma_i : p_i := p_i - 1; p_{i+1} := p_{i+1} + 1;$
- The "problem" of $A_i = 1$.
  - Implies that $a_i, a_{i+1} = 1$. (Since $A_i = a_i + a_{i+1} - 1$)
  - I.e., cells i and $i+1$ are frozen in place.
- If two or more $A_i = 1$ then underlying graph is disconnected; otherwise it is connected.
- The graph is denoted $G([a_1, \ldots, a_k])$ or $G(\langle A_1, \ldots, A_{k-1} \rangle)$.
- An interesting case occurs when $A_{k-1} = 1$ and $A_i > 1$, for $i = 1, 2, \ldots, k-2$: the *right-frozen* case.
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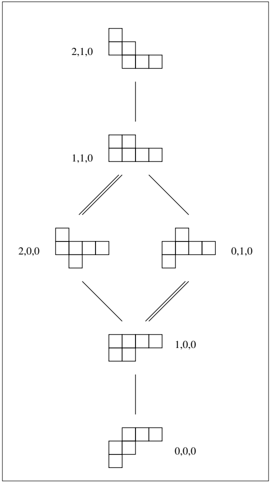
- The graph is denoted $G([a_1, \ldots, a_k])$ or $G(\langle A_1, \ldots, A_{k-1} \rangle)$.
- An interesting case occurs when $A_{k-1} = 1$ and $A_i > 1$, for $i = 1, 2, \ldots, k - 2$: the *right-frozen* case.
- *Free* = neither side frozen.



Lemma

If k is even, then $G([a_1, \ldots, a_k])$ is bipartite.

Proof.

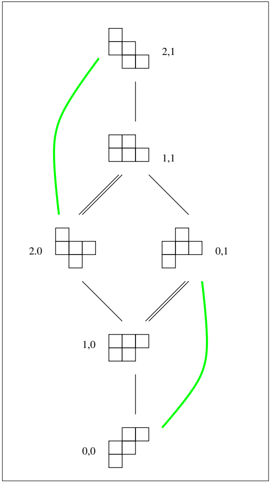
Define partite sets according to the parity of

$$\sum_{j \text{ odd}} p_j \quad \left(= \sum_j j p_j \bmod 2\right).$$

□

Example

$G([2, 2, 1, 1])$ is bipartite.



Lemma

If $k > 3$ is odd, then $G([a_1, \ldots, a_k])$ is bipartite iff $A_1 = 1$ or $A_{k-1} = 1$.

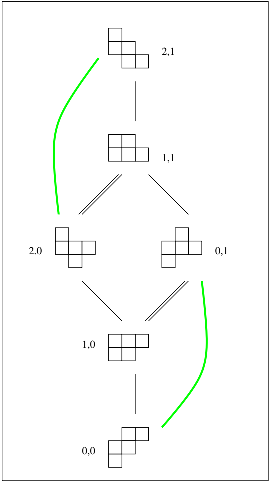
Proof.

Odd cycle: $\tau_1, \sigma_1, \ldots, \sigma_{k-2}, \tau_{k-1}^{-1}$.

□

Example

$G([2, 2, 1])$ is not bipartite.



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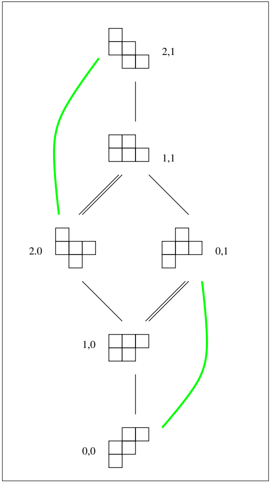

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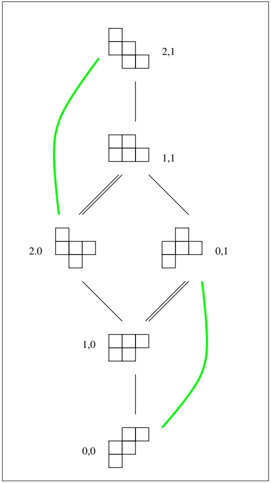

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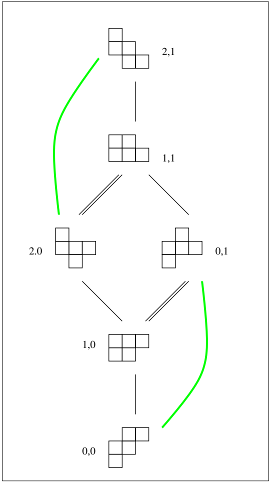

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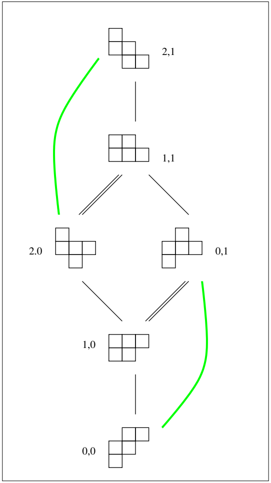

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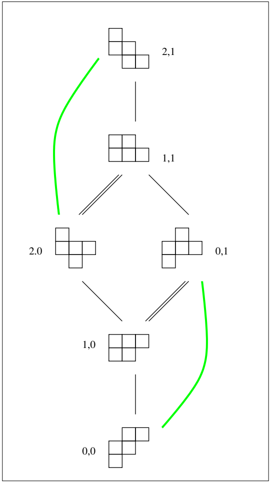

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□

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$G([2, 2, 1])$ is not bipartite.

Lemma

If $k > 2$ is even, then $G(\mathbf{a})$ has no Hamilton path if A_{2i+1} is odd for all i, unless $a_2 = a_3 = \cdots = a_{k-1} = 1$.

Proof.

Define a sign-reversing involution... If all A_{2i+1} are odd, then parity difference is

$$d(\mathbf{A}) = \prod_{j \text{ even}} A_j.$$

□

Example

$G(\langle 3,2,3,2,3 \rangle)$ has no H-path.
$G(\langle 3,1,1,1,3 \rangle)$ is a 3 by 3 grid and has a H-path.
Conjecture: There is a H-path if k even and some A_{2i+1} is even.

Theorem

If $G(\langle A_1, A_2, \ldots, A_{k-1}\rangle)$ has a Hamilton path H and BC is even and $BC \neq 6$, then $G(\langle B, A_1, A_2, \ldots, A_{k-1}, C\rangle)$ has a Hamilton path H'.

Proof.

Convert each edge of H into a path of BC vertices in H'. The structure of B, C is a B by C grid graph. Need to be careful about the τ moves in H.... Need to show the existence of particular types of H-cycles in grid graphs.... □

Theorem

If **A** *has the form $(N_{>3})^{0|1}(O_{>3}N_{>1})^*(N)^{0|1}E$ or it's reverse, then there is a 1-move Gray code for* **A**

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Add pairs of columns on the left... □

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Converting $G(2, 2)$ into $G(2, 2, 2, 2)$.

0	0	1	1
0	1	0	1

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?	?	?	?	
0	0	0	0	0
0	0	0	0	1
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0	1	1	0	0	?	?	?	
0	0	0	0	0	0	0	0	1
0	0	0	0	1	1	1	1	0
0	0	1	1	0	?	?	?	

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0	0	0	0	0	0	0	0	1	1	1	1	1
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0	0	1	1	0	1	1	0	0	?	?	?	

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0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	1	1	0	1	1	0	0	0	1	1	0	?	?	?

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0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
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Right-frozen case – a poset

- Operations are τ_1 and $\sigma_1, \sigma_2, \ldots$.
- Underlying poset $G(\langle A_1, A_2, \ldots, A_{k-1}, 1\rangle)$ orders **p**-sequences: $(p_1, p_2, \ldots, p_{k-1}) \leq (q_1, q_2, \ldots, q_{k-1})$ iff

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$$r(\mathbf{p}) = \sum_{j=1}^{k-1} j p_j .$$

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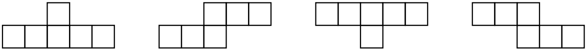

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This follows from the results of Savage, Shields, and West. □

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$$\sum_{j=1}^{k} j(A_j - 1) = \sum_{j=1}^{k} jA'_j. \tag{1}$$

Proof.

Note that $00\cdots00$ and $A'_1 A'_2 \cdots A'_k 0$ are pendant vertices in a $A_1 A_2 \cdots A_k$ vertex bipartite graph G and that their distance from each other is precisely (1). □

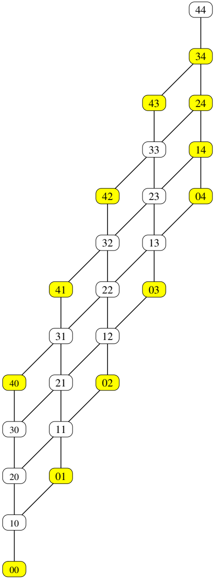
Lemma

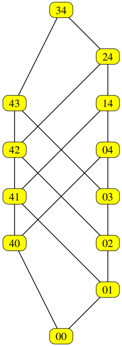
There is no Hamilton path in $G(\langle A_1, A_2, \ldots, A_k, 1\rangle)$ *if (1) has the same parity as* $A_1 A_2 \cdots A_k$.

$$\sum_{j=1}^{k} j(A_j - 1) = \sum_{j=1}^{k} jA'_j. \tag{1}$$

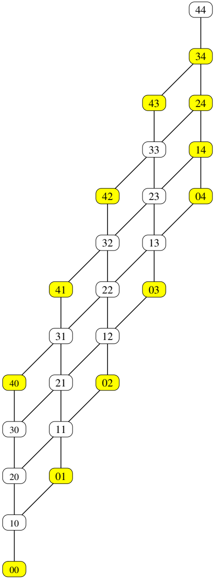
Proof.

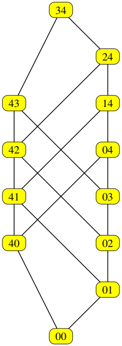
Note that $00 \cdots 00$ and $A'_1 A'_2 \cdots A'_k 0$ are pendant vertices in a $A_1 A_2 \cdots A_k$ vertex bipartite graph G and that their distance from each other is precisely (1). □





- The Poset $G(\langle 5, 5, 1\rangle)$. Join irreducibles in yellow.
- The subposet of join-irreducibles.
- Both are self-dual.
- Note: $G(\langle 5, 5, 1\rangle)$ not rank-unimodal.





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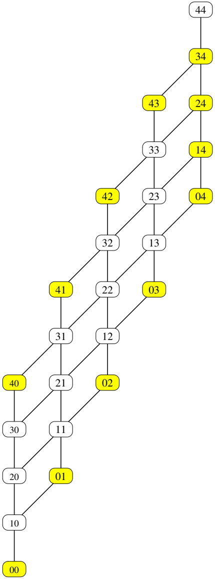

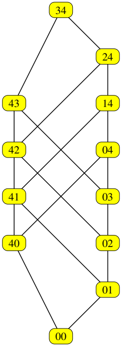

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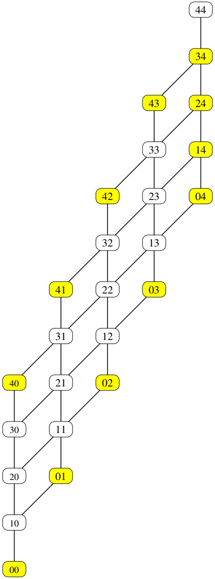

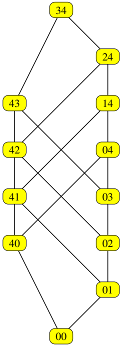

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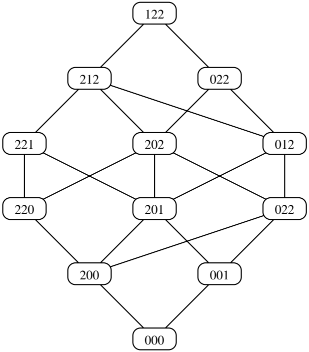

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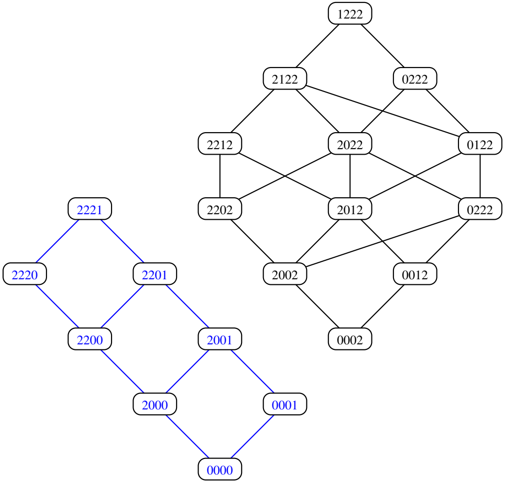

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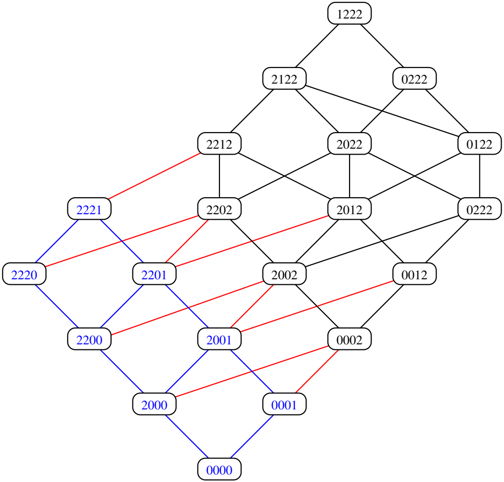

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More on unimodality

- Let $c(S)$ be the smallest value of k for which $P(S^i)$ is rank unimodal for all $i \geq k$.
- **Example** What is $c(2,3)$? The rank gf is $(1+z)(1+z^2+z^4) = 1+z+z^2+z^3+z^4+z^5$.
- Ranks of $2,3,2$:

1	1	1	1	1	1			
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1	1	1	2	2	2	1	1	1

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1	1	1	2	2	2	1	1	1								
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- Continuing, $P((2,3)^5)$ is not unimodal, but $P((2,3)^k)$ appears to be unimodal for $k \geq 6$ (checked up to $k = 200$).

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$c(n,m)$

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	1	**6**	1	7	1	5	1	7	1	5	1	5	1	5	1	5
3	7	6	8	8	8	a	8	a	9	a	b	a	b	a	c	c
4	4	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	6	1	3	4	6	5	6	5	6	6	7	6	7	6	7	6
6	1	1	1	3	1	1	1	3	1	3	1	3	1	3	1	3
7	3	1	1	2	4	4	4	4	4	4	5	5	5	6	5	6
8	3	1	1	3	1	3	1	1	1	3	1	1	1	1	1	1
9	4	1	1	1	3	3	3	3	4	4	4	4	4	4	4	4
10	3	1	1	1	1	1	1	3	1	1	1	3	1	1	1	3
11	3	1	3	1	1	2	4	4	4	4	4	4	4	4	4	4
12	1	1	1	1	1	3	1	3	1	3	1	3	1	3	1	1
13	4	1	3	1	3	1	2	2	3	3	3	3	3	3	4	4
14	4	1	3	3	1	3	1	3	1	3	1	3	1	1	1	3
15	1	4	4	4	1	4	4	2	4	4	4	4	4	4	4	4
16	4	1	3	3	1	1	1	3	1	3	1	3	1	3	1	3
17	4	1	3	3	1	1	3	1	3	2	3	3	3	3	3	3
18	4	1	1	1	1	1	1	3	1	3	1	3	1	3	1	3
19	4	1	4	3	1	1	1	1	1	2	3	2	3	3	3	3
20	3	1	3	3	1	3	1	1	1	3	1	3	1	3	1	3

Remarks about $c(m, n)$

- Only the 2,2 entry is proven.
- Most entries are checked out to $k = 30$. The numbers are huge; i.e., the underlying poset has $(nm)^{30}$ elements.
- Some conjectures:
  - For odd m, $\lim_{n \to \infty} c(m, n) = c_m$. In particular, $c_3 = 16$ (the value 16 first occurs for $k = 90$).
  - For even m with an odd factor, for large enough n, if n even then $c(m, n) = 1$ and if n is odd then $c(m, n) = 3$.
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- Is the Gray code necessary condition sufficient?
- Remove the $BC \neq 6$ from a previous theorem.
- Gray codes for general column-convex polyominoes?
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Thanks for still being here!

And please let me know if you have seen that poset before...

And on to the Venn diagrams.

Definition - Venn diagram

- Made from simple closed curves $C_1, C_2, \ldots, C_n$.
- Infinite intersection not allowed (usually, but not here!).
- Let X_i denote the interior or the exterior of the curve C_i and consider $X_1 \cap X_2 \cap \cdots \cap X_n$.
- *Euler diagram* if each such intersection is connected.
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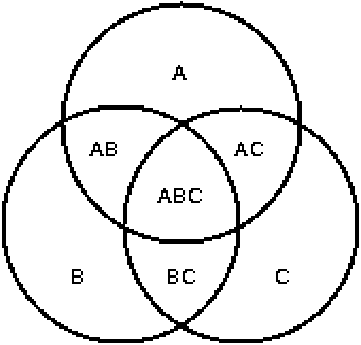

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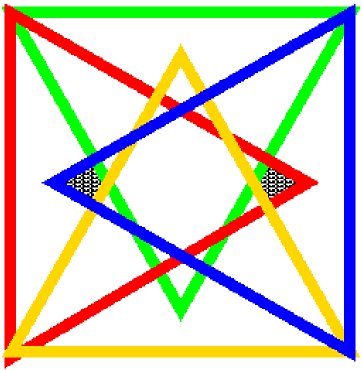
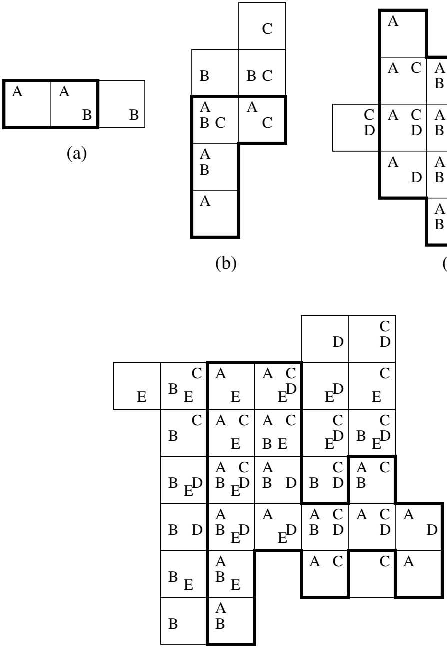
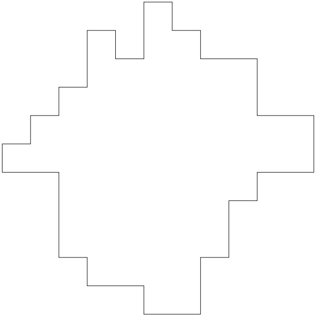


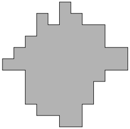
"polyVenn" diagrams

- Minimum area Venn diagrams.
- Curves are the boundaries of a polyomino.
- Each interior/exterior intersection is a square.
  - Total bounded area is $2^n - 1$.
  - Each polyomino is a 2^{n-1}-omino.
- Introduced by Mark Thompson on a recreational math web page.

PolyVenn diagram with congruent curves







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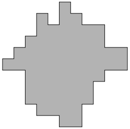

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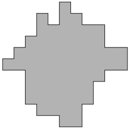

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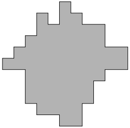

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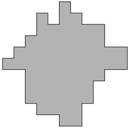

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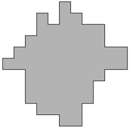

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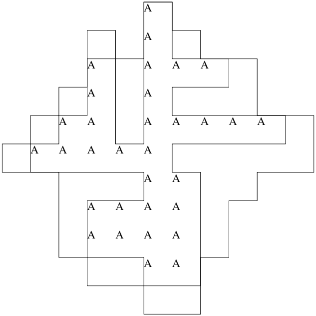

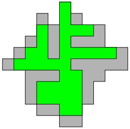
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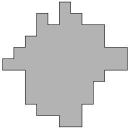
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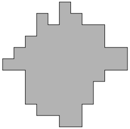
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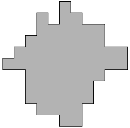
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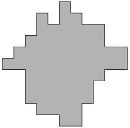
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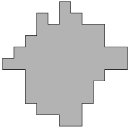
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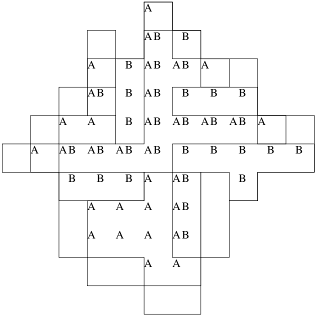
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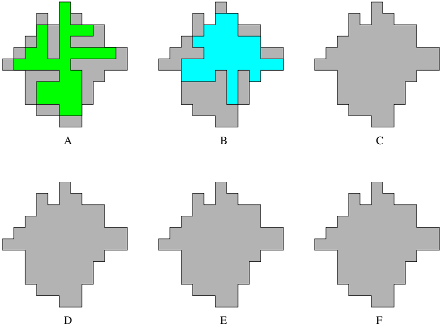
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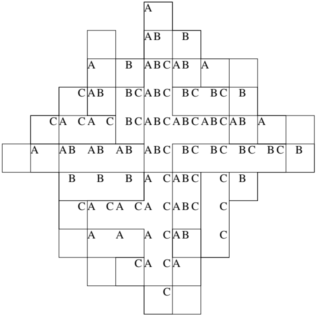
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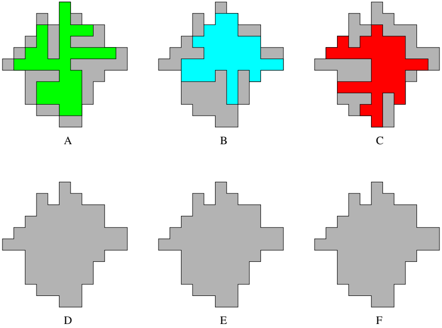
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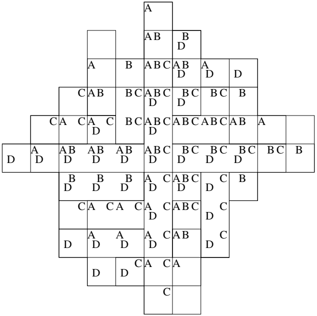
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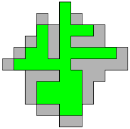

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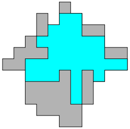

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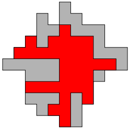

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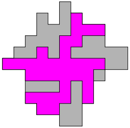

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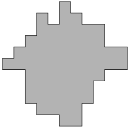

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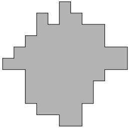

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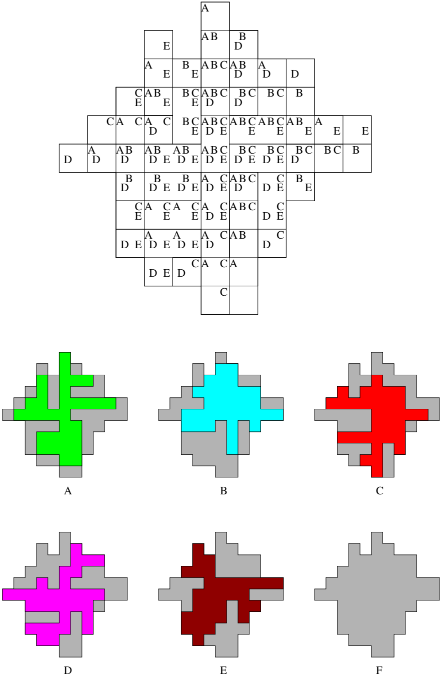
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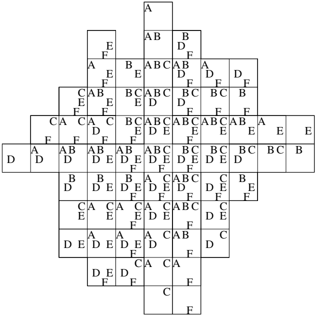

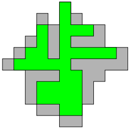

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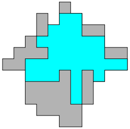

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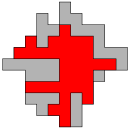

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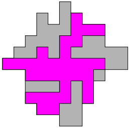

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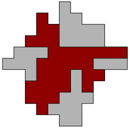

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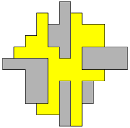

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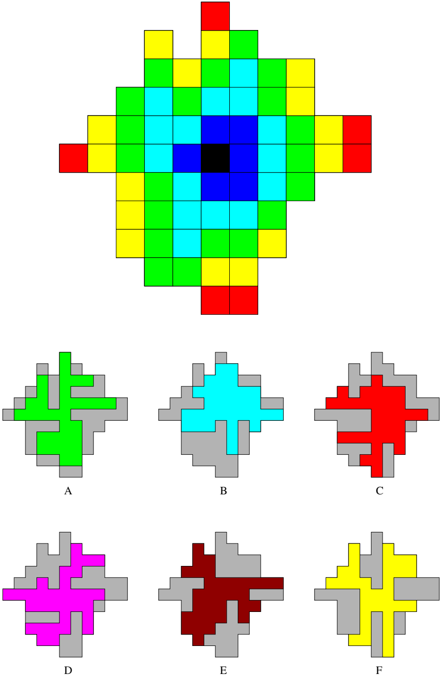
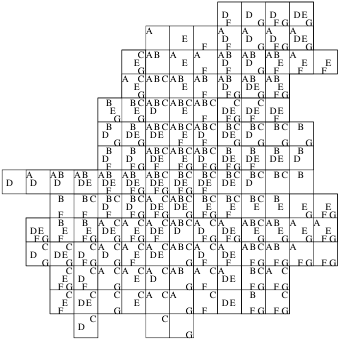
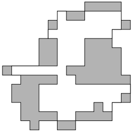


Top figure colored by cardinality of the underlying set. Red = 1, yellow = 2, etc.





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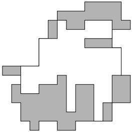
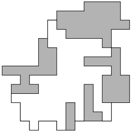


B

$n = 7.$



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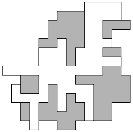

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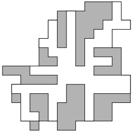

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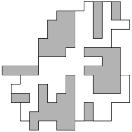

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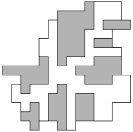
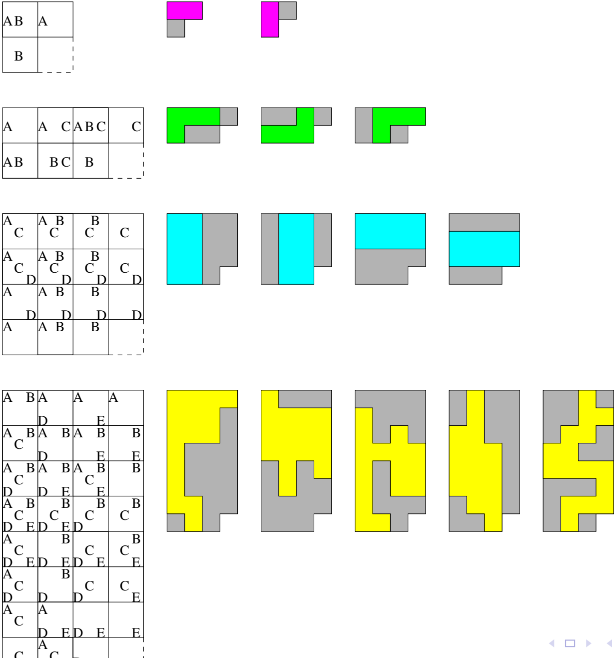
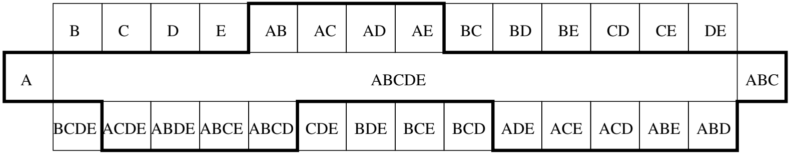


G

In a minimum bounding box



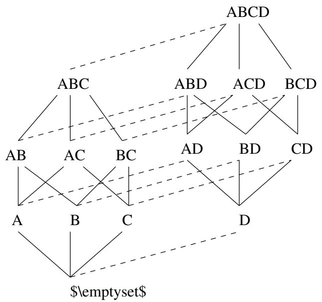
A naïve way to construct a polyVenn diagram for any n.



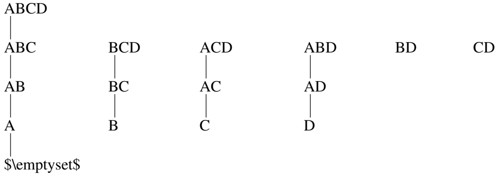
- A 5-Venn diagram with curve A highlighted.
- Area in general is $\frac{3}{2}2^n$.

Symmetric chain decomposition of the Boolean lattice

.



(a)



(b)

- A partition of all 2^n subsets into chains of the form

$$x_1 \subset x_2 \subset \cdots \subset x_t$$

where $|x_i| = n - |x_{t-i+1}|$ and $|x_i| = |x_{i-1} - 1|$.

- Various algorithms known:
  - De Bruijn, van Ebbenhorst Tengbergen, and Kruyswijk.
  - Aigner.

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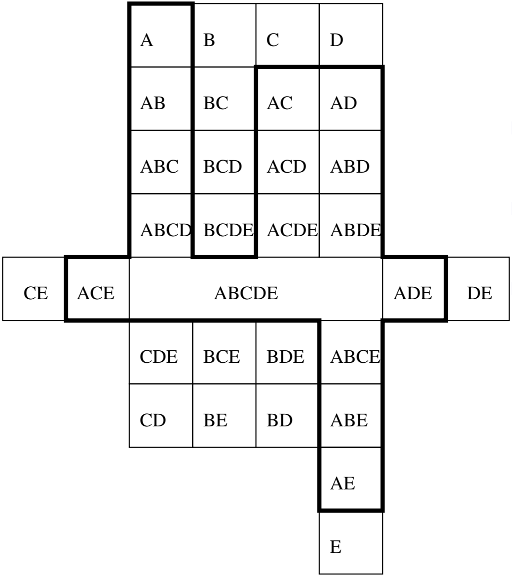

- Another 5-Venn diagram with curve A highlighted.
- Based on symmetric chain decomposition.
- Area is $2^n + \binom{n}{n/2}$.

- Are there congruent n-polyVenns for $n \geq 6$? We saw earlier that they exist for $n = 2, 3, 4, 5$.
- Is there a 5-polyVenn whose curves are convex polyominoes?
- Are there minimum bounding box n-polyVenns for $n \geq 6$? We saw earlier that they exist for $n = 2, 3, 4, 5$.
- Are there minimum area n-polyVenns for $n \geq 8$? We saw earlier examples for $n = 6, 7$.
- One problem for which we have not attempted solutions is the construction of n-polyVenns that fill an $w \times h$ box, where $wh = 2^n - 1$. Of course, a necessary condition is that $2^n - 1$ not be a Mersenne prime. For example, is there are 4-polyVenn that fits in a 3×5 rectangle or a 6-polyVenn that fits in a 7×9 or 3×27 rectangle?

Thanks for coming!