Every Simple Venn Diagram is Hamiltonian

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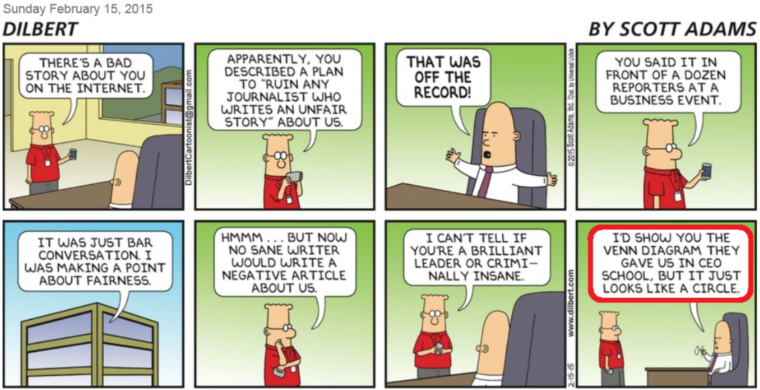
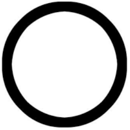
CanaDAM 2015, Saskatoon

Venn diagram examples; famous and otherwise ($n = 1$).



$n =$ number of curves $= 1$



Venn diagram examples; famous and otherwise ($n = 2$).

Mitt Romney doesn't understand Venn diagrams

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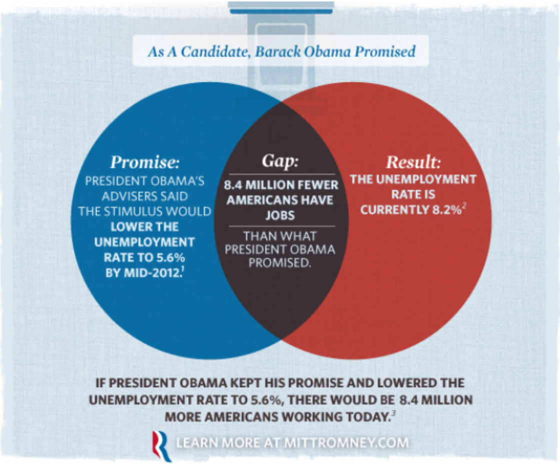

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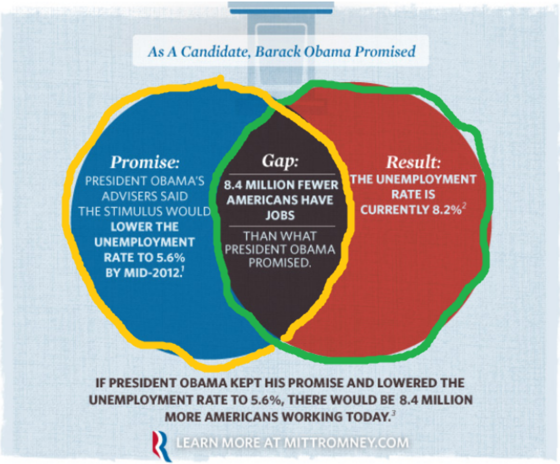

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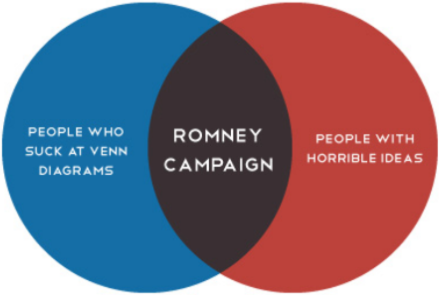

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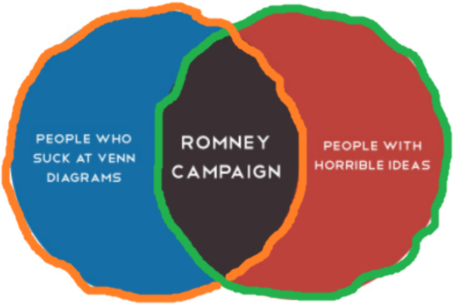

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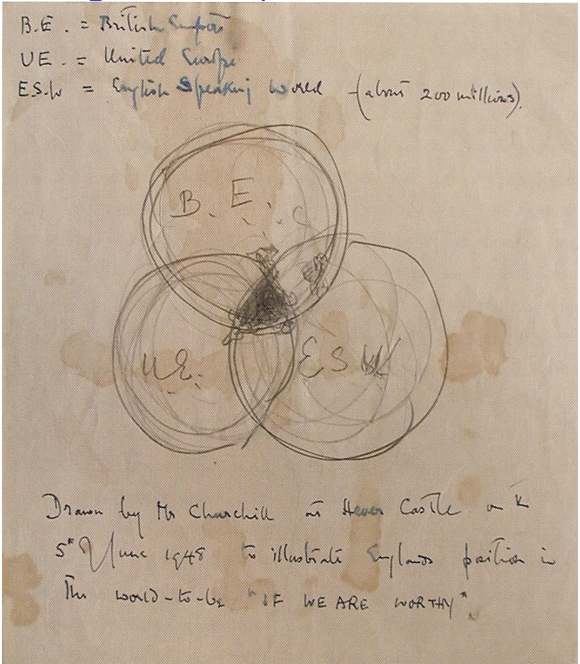
Venn diagram examples; famous and otherwise ($n = 3, 4$).



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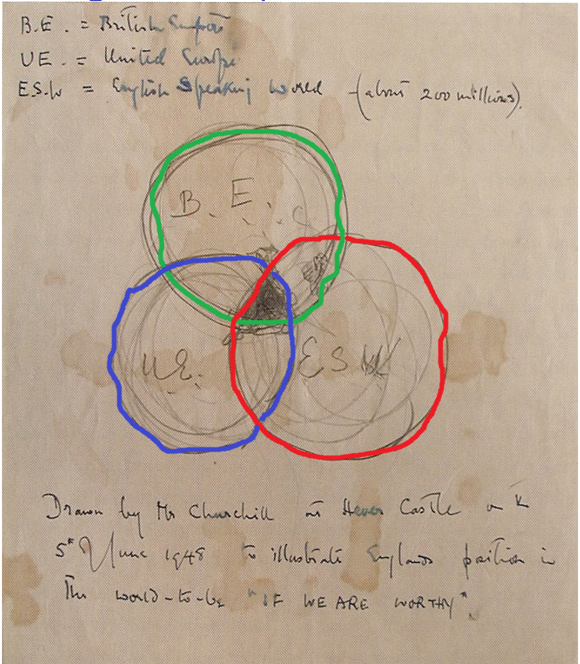

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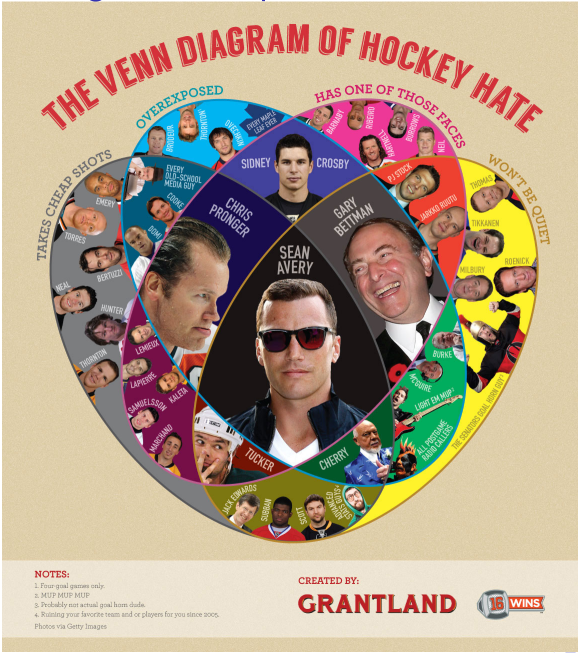
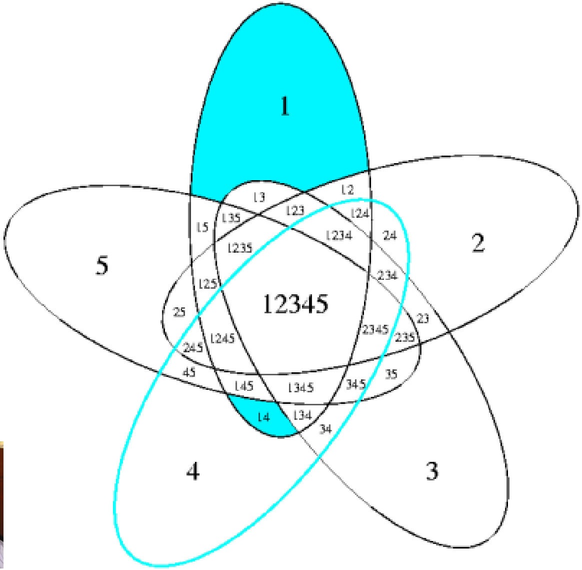


An irreducible Venn diagram ($n = 5$)



What is a Venn diagram?

- Made from simple closed curves $C_1, C_2, \ldots, C_n$.
- Only finitely many intersections.
- Each such intersection is transverse (no "kissing").
- Let X_i denote the interior or the exterior of the curve C_i and consider the 2^n intersections $X_1 \cap X_2 \cap \cdots \cap X_n$.
- *Euler diagram* if each such intersection is connected.
- *Venn diagram* if Euler and no intersection is empty.
- *Independent family* if no intersection is empty.

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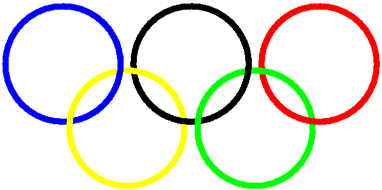

Euler but not Venn

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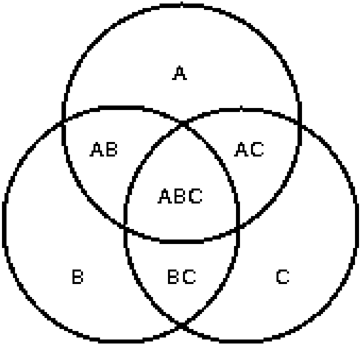

Venn (and Euler)

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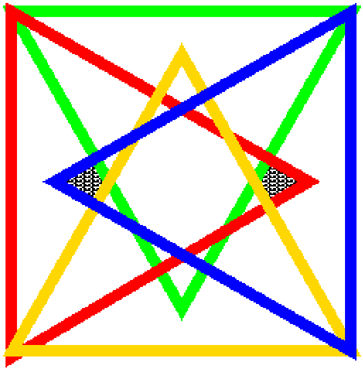

Neither Venn nor Euler

Winkler's conjecture

- An n-Venn diagram is *reducible* if there is some curve whose removal leaves an $(n-1)$-Venn diagram.
- An n-Venn diagram is *extendible* if the addition of some curve results in an $(n+1)$-Venn diagram.
- Not every Venn diagram is reducible. Every reducible diagram is extendible.
- **Conjecture:** Every *simple* n-Venn diagram is extendible to a *simple* $(n+1)$-Venn diagram.
- Reference: Peter Winkler, Venn diagrams: Some observations and an open problem, Congressus Numerantium, 45 (1984) 267–274.
- The conjecture is true if the simplicity condition is removed (Chilakamarri, Hamburger, and Pippert (1996)).
- The conjecture is true if $n \leq 5$. Determined by Bultena; there are 20 non-isomorphic (spherical) diagrams to check.

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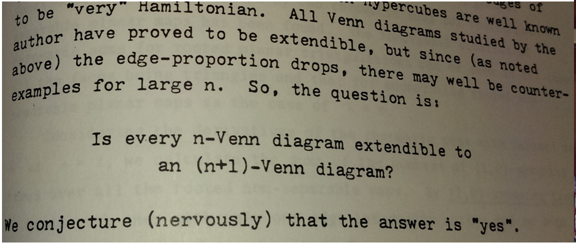
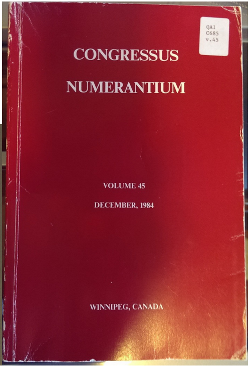
Winkler's conjecture



to be "very" Hamiltonian. All Venn diagrams studied by the ... hypercubes are well known
author have proved to be extendible, but since (as noted
above) the edge-proportion drops, there may well be counter-
examples for large n. So, the question is:

Is every n-Venn diagram extendible to
an (n+1)-Venn diagram?

We conjecture (nervously) that the answer is "yes".

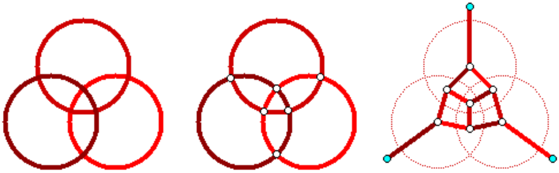


Puzzled
Where Sets Meet (Venn Diagrams)

Welcome to three new puzzles. Solutions to the first two will be published next month; the third is as yet unsolved.

3. Prove *or disprove* that to any Venn diagram of order n another curve can be added, making it a Venn diagram of order $n+1$; remember, only simple crossings allowed.

Wishes and Reality

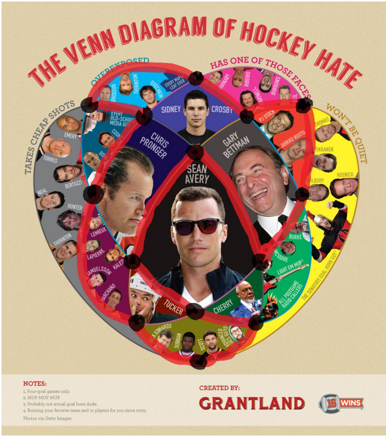


The Venn Diagram, C

The Venn Diagram V(C) as an edge labelled Graph.

The Venn Dual, a 3−cube, D(C) The cyan vertices are identified.

- **What we want to prove:** The dual of every simple Venn diagram is Hamiltonian.
- **What we can prove:** Every simple Venn diagram is Hamiltonian. Here intersection points are vertices and curve segments between vertices are edges.
- **Is it progress?** Good question. Remains to be seen.



Our result

Winkler conjecture

Proof Strategy

- Previously, it was know that any simple Venn graph is 3-connected (Kiran B. Chilakamarri, Peter Hamburger and Raymond E. Pippert, *Analysis of Venn diagrams using cycles in graphs*, Geometriae Dedicata, 82 (2000) 193–223).
- But if we can show that the graph is 4-connected then Tutte's theorem applies.

Theorem (Tutte, 1956): Every 4-connected planar graph is hamiltonian.

-
- Alternate characterization of k-connectivity: **Theorem (Menger):** A graph is k-connected if and only if between every pair of distinct vertices there are at least k pairwise vertex-disjoint paths.

A useful (new?)[1] lemma

Lemma: A connected graph is k-connected if and only if for every pair of vertices u,v *at distance 2* there are at least k vertex disjoint paths between v and v.

Proof: $\Rightarrow$ is immediate.

- Minimum cutset X and vertex $x \in X$.
- Graph $G - X + x$ is connected, x has neighbors u, v on each side.
- k disjoint paths from u to v, each hits at least one vertex in X.
- Thus $|X| \geq k$; i.e., the minimum cutset size is at least k.

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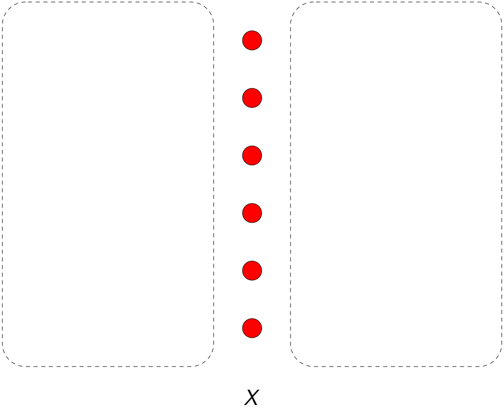

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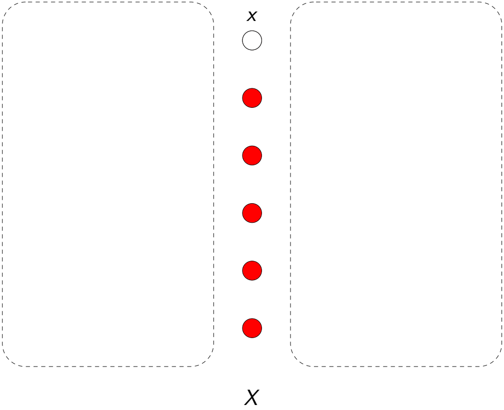

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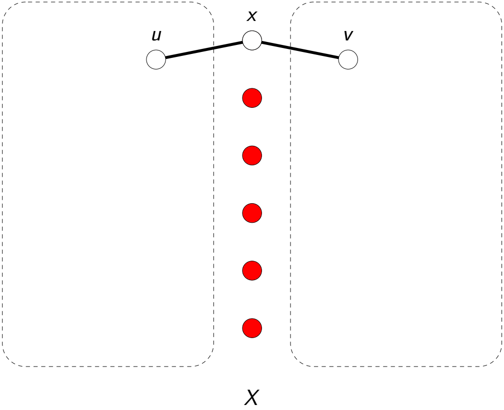

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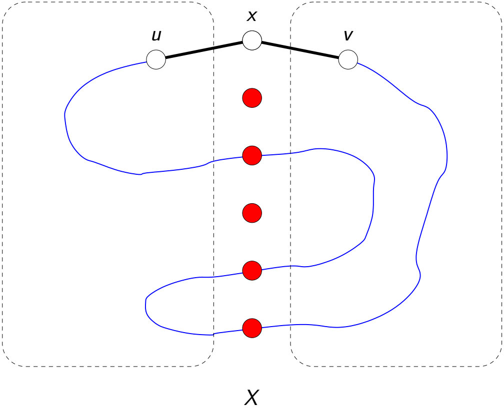

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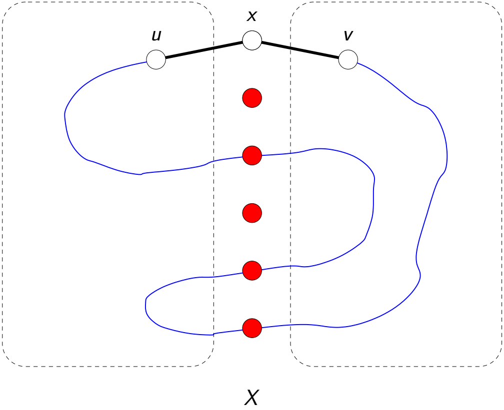

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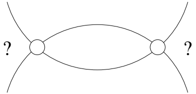
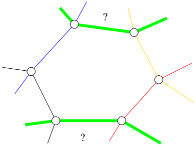
The proof of 4-connectivity

- We only use two properties of Venn diagrams ($n > 2$):
  - There are no 2-faces.

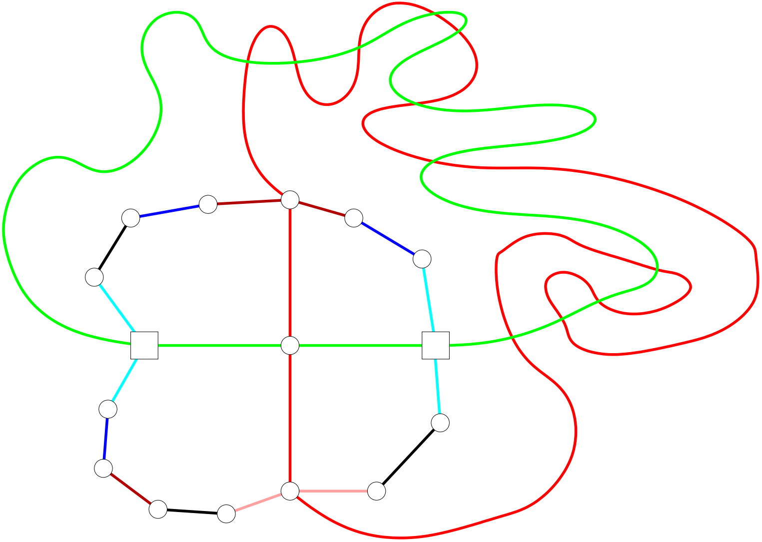


  - No face has 2 (or more) instances of the same curve.



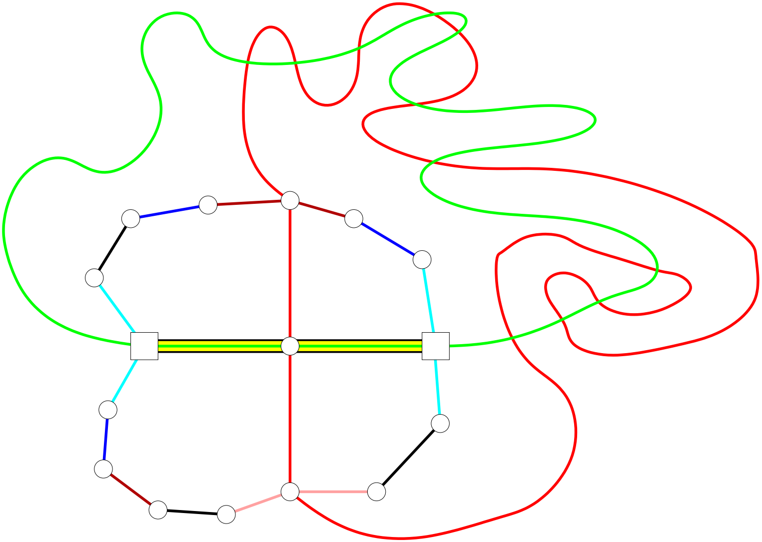
  - The second condition implies the first.
- Let □ and □ be any two vertices at distance 2. There are 2 cases.
- The two edges on the path are from the same (green) curve.
- The two edges on the path are from different curves (red & green).

Case 1: □, □ are on the same curve



Case 1:Both of □ and □ are on the same (green) curve.

Case 1: □, □ are on the same curve



Path 1: Use the length 2 green path.

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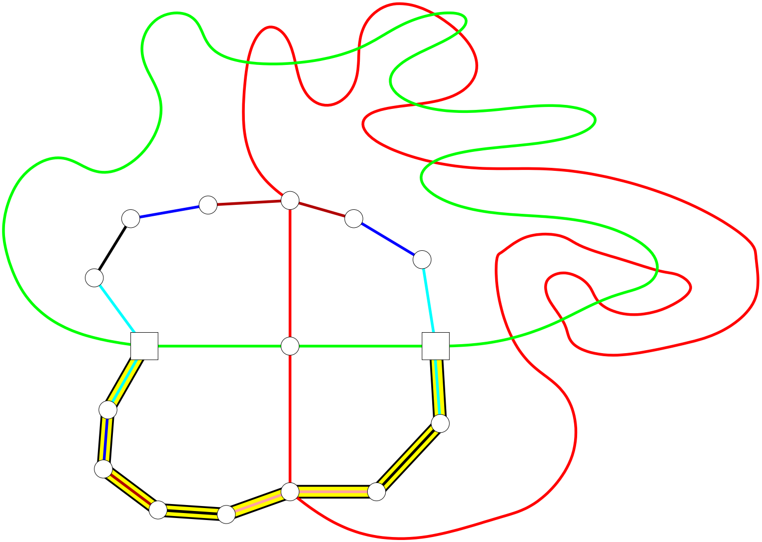
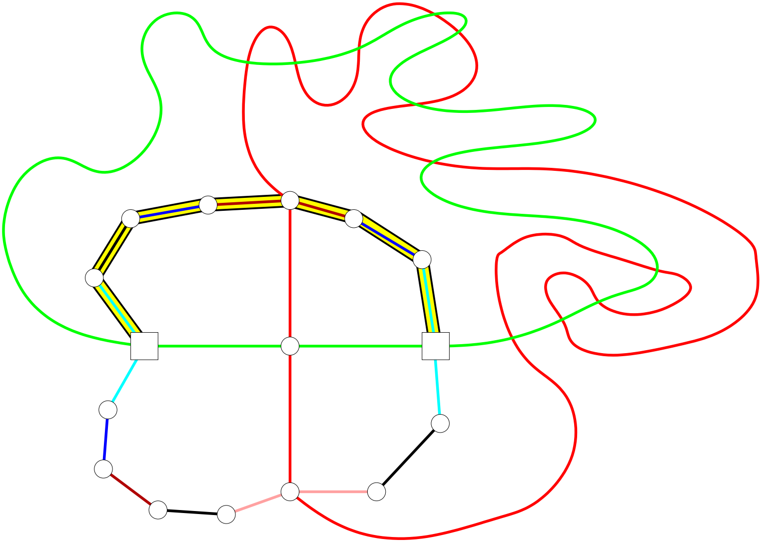


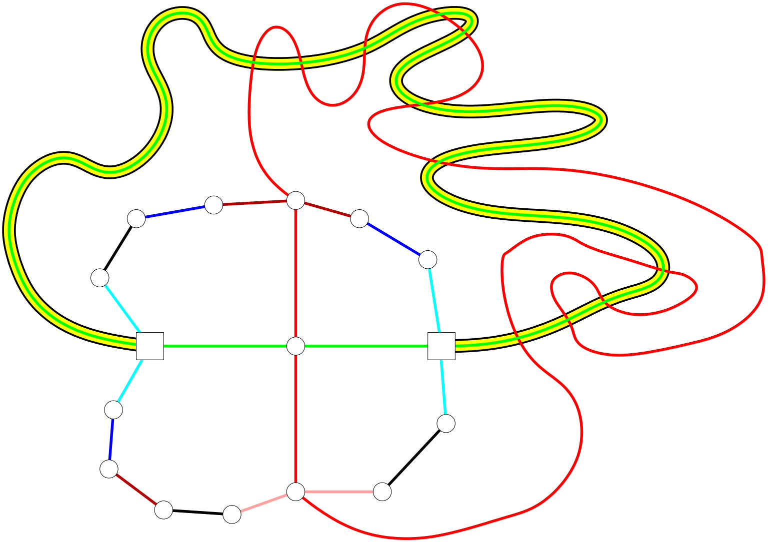
Path 2: Use the edges on two adjacent faces.

Case 1: □, □ are on the same curve



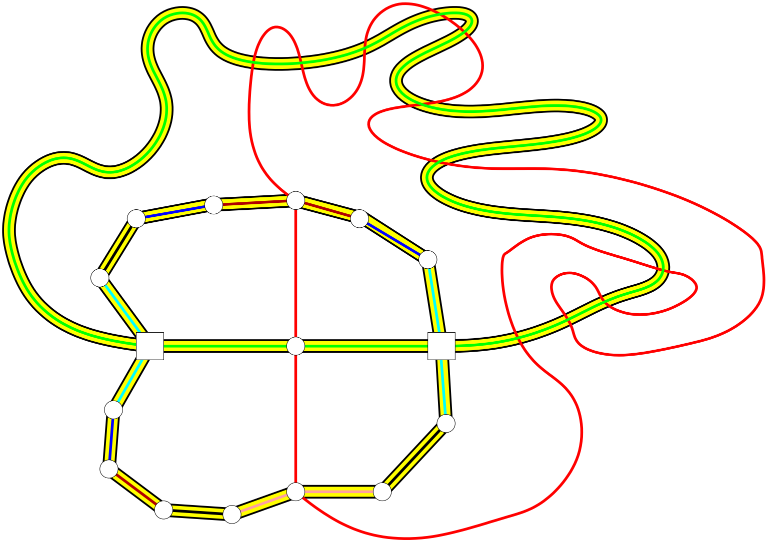
Path 3: Use the edges on the other two adjacent faces.

Case 1: □, □ are on the same curve



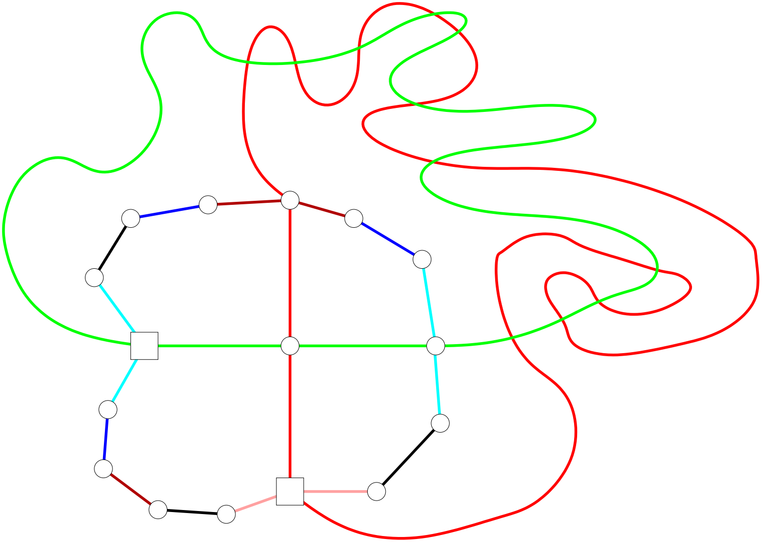
Path 4: Use the unused green edges.

Case 1: □, □ are on the same curve



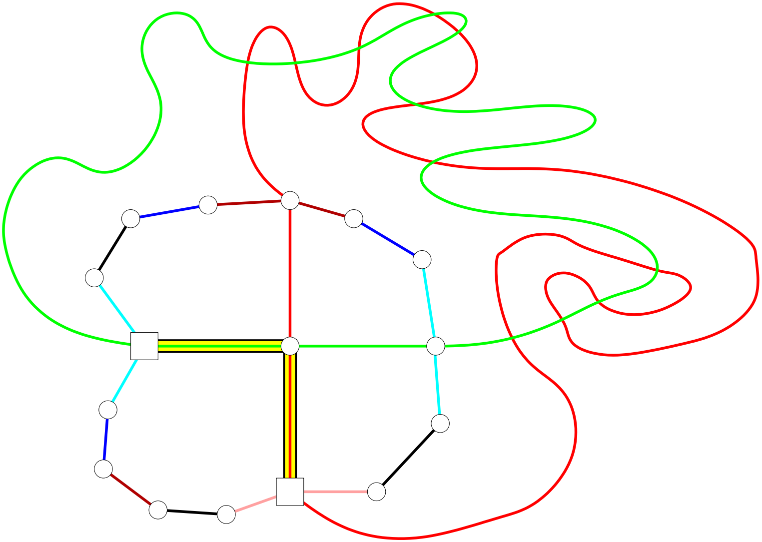
All four paths shown at once.

Case 1: □, □ are on the different curves



Case 1: Vertices □ and □ are on different curves.

Case 1: □, □ are on the different curves



Path 1: Use the length 2 path between □ and □.

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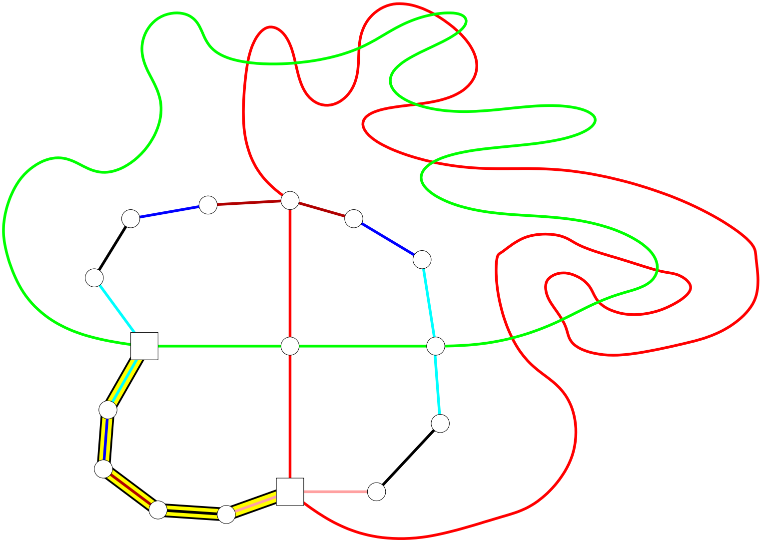
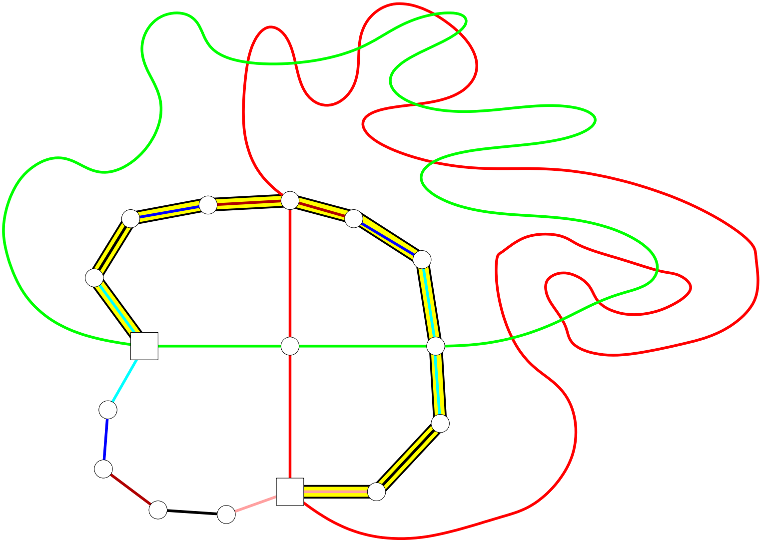


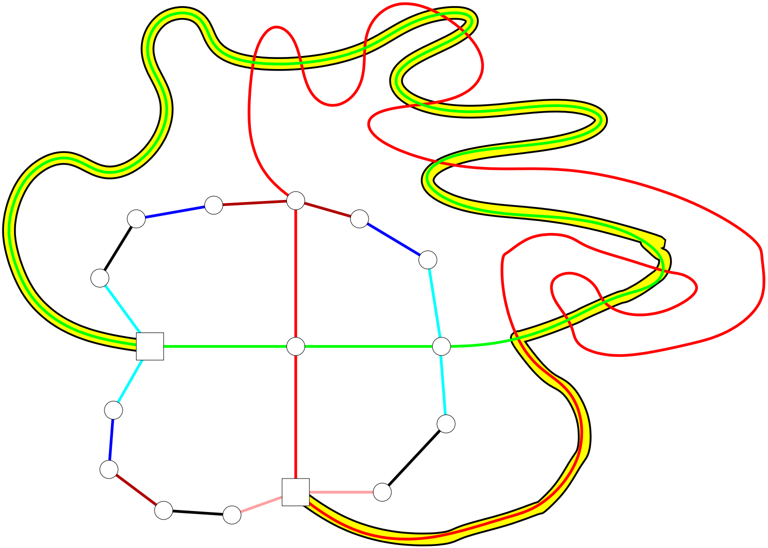
Path 2: Use the edges on the common face.

Case 1: □, □ are on the different curves



Path 3: Use the edges on the other three faces.

Case 1: □, □ are on the different curves



Path 4: Use green path to last red intersection, then red edges.

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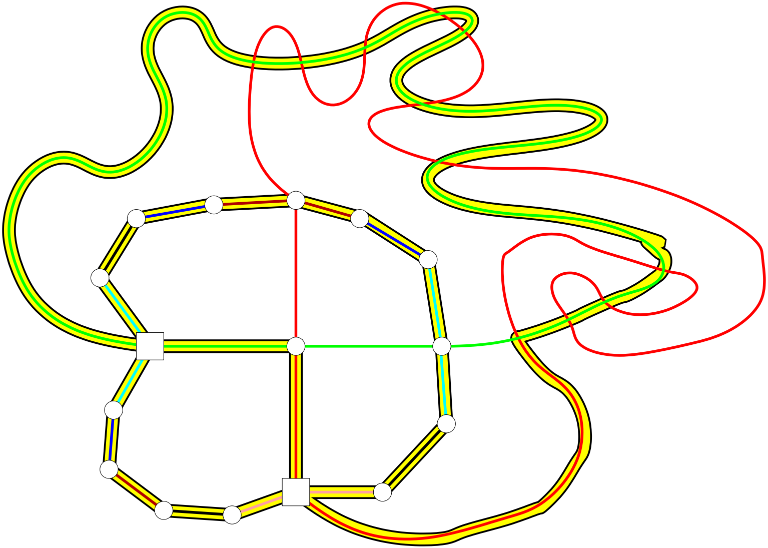

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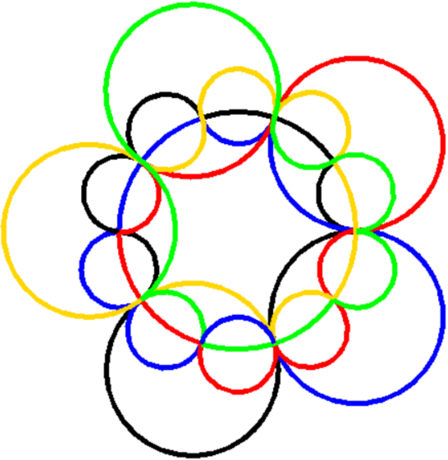
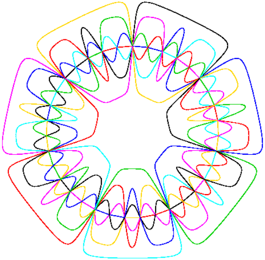
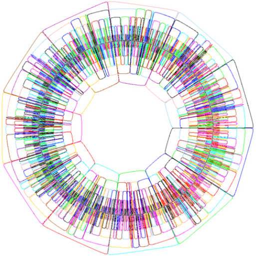
What about non-simple Venn diagrams?

They are only 2-connected in general:

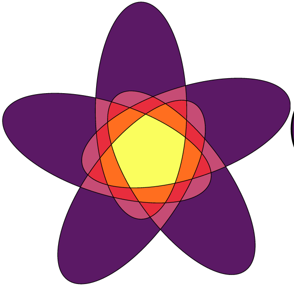


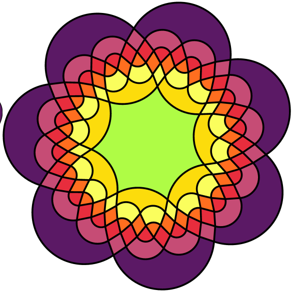




Examples of a general family on prime numbers of curves.

Tutte's Theorem for Winkler's conjecture?



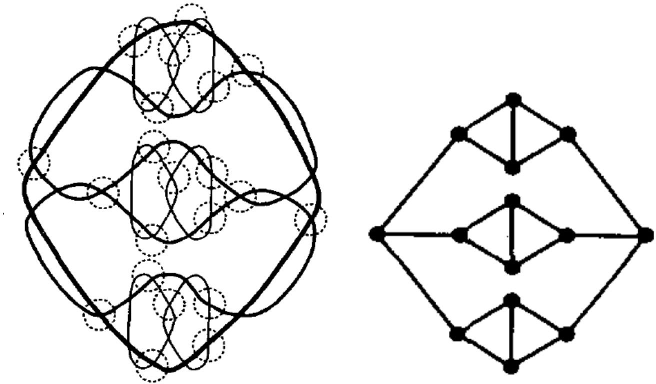


Problem: Venn diagram duals are only 3-connected in general, because Venn diagrams have 3-faces. In fact

Theorem

For $n \geq 3$, any n-Venn diagram has at least 8 3-faces.

A 3-connected non-Hamilton collection of curves



Iwamoto & Touissant (1994) *Finding Hamiltonian circuits in arrangements of Jordan curves is NP-complete*.

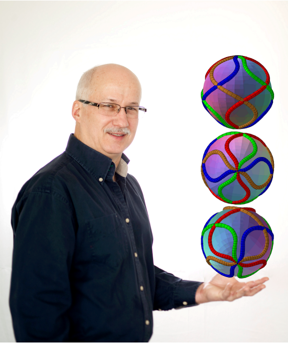
Open problems

- Is every *non-simple* Venn graph Hamiltonian?
- What is the internal connectivity of a Venn dual? (KY)

Easier versions of Winkler's conjecture:

- Does every Venn diagram dual have a perfect matching?
- Is every monotone Venn diagram extendible? Monotone = drawable with all curves convex.
- Is the prism ($G \times e$) of every Venn diagram dual hamiltonian?

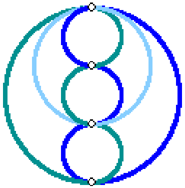
The End



Thanks for coming.
Any questions?

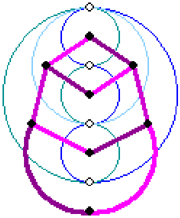
Theorem: Every Venn diagram is extendible.

Proof: Form radual graph. Apply Whitney's theorem.



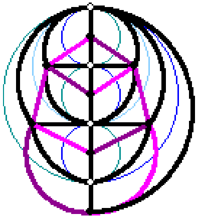
The Venn Diagram #3.4

Blueish edges



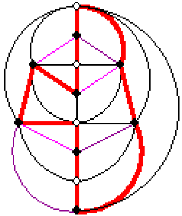
The Dual Graph

Magenta edges

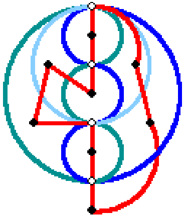


The Radual Graph

Black and magenta edges



Hamilton Cycle



Extended Venn Diagram

Red and blueish edges