Domino Tatami Covering is NP-complete

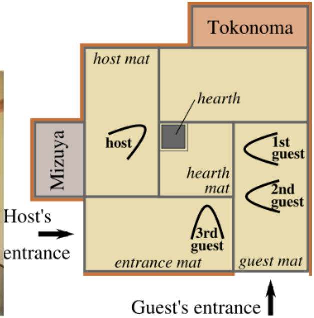
Alejandro Erickson[†] and Frank Ruskey

University of Victoria
IWOCA 2013, Rouen, France
Proceedings: paper_91.pdf

July 10-12, 2013

Japanese Tatami mats

Traditional Japanese floor mats made of soft woven straw.



A 17th Century layout rule:
No four mats may meet.

No four dominoes (mats) may meet

Tatami coverings of rectangles were considered by Mitsuyoshi Yoshida, and Don Knuth (about 370 years later).

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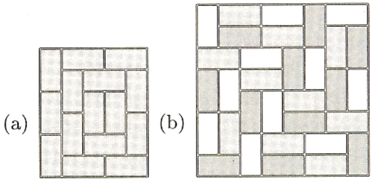
Find all domino coverings of a chessboard that are also tatami tilings.

Fig. 29. Two nice examples:
(a) A 17th-century tatami tiling;
(b) a tricolored domino covering.



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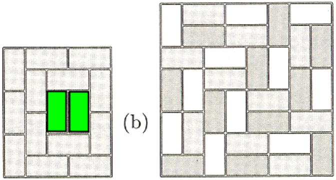
(a) (b)



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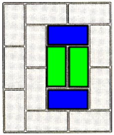
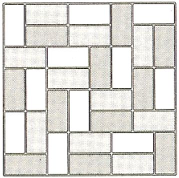


(b)



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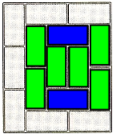

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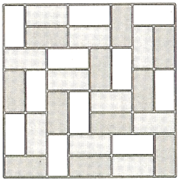

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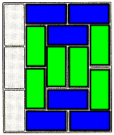
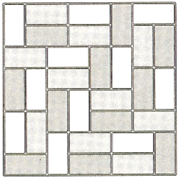
(a) 

(b) 

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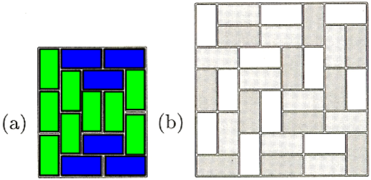

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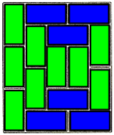

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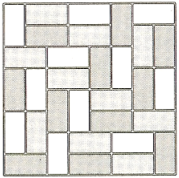
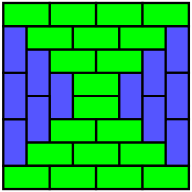
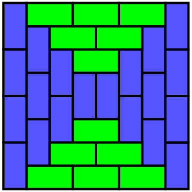


Coverings of the chessboard

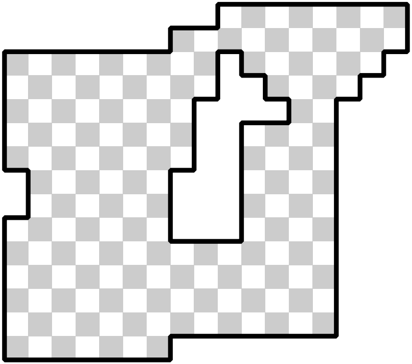
There are exactly two

Generalized by Ruskey, Woodcock, 2009, using Hickerson's decomposition.





Domino Tatami Covering



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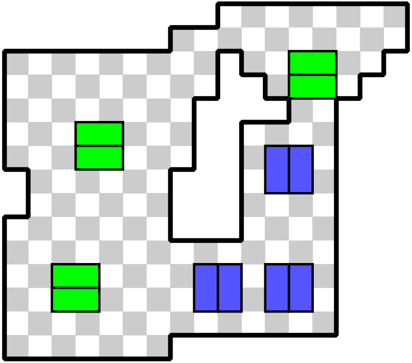

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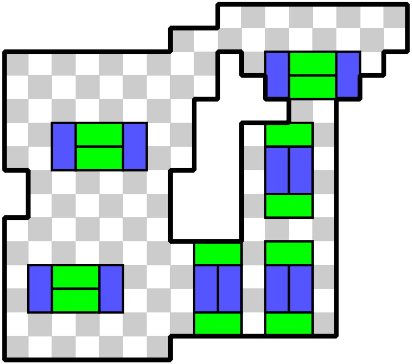

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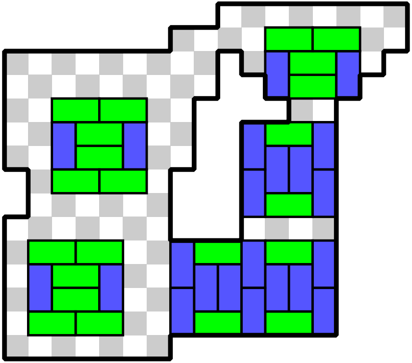

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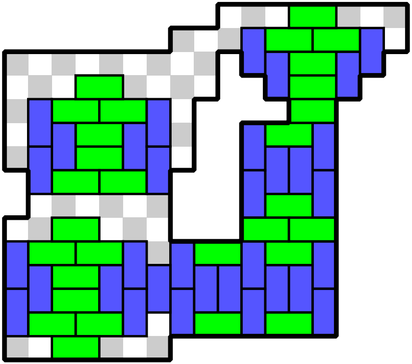

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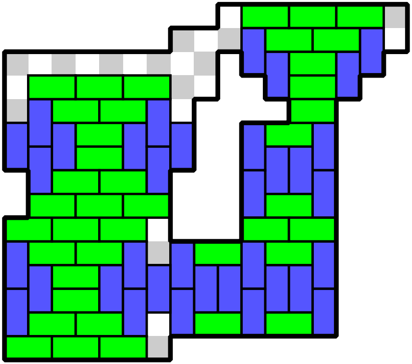

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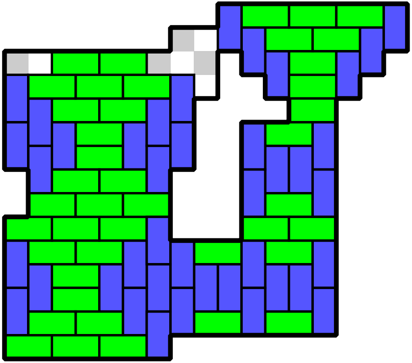

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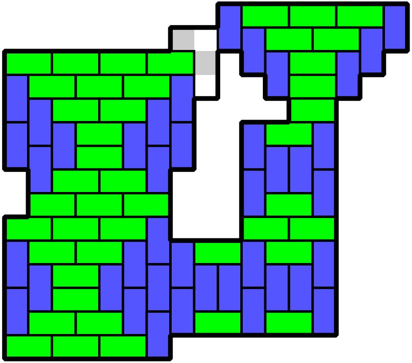

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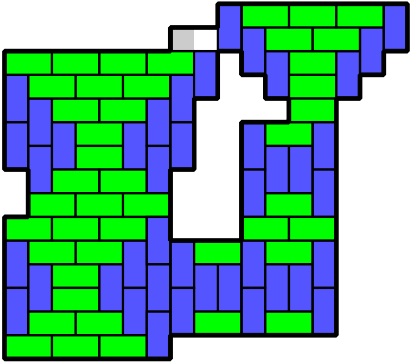

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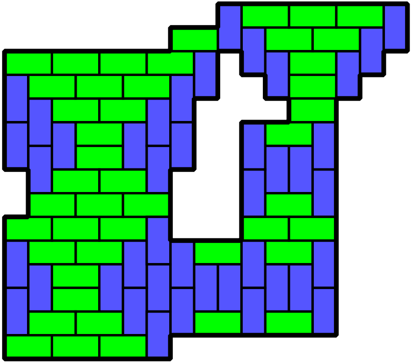

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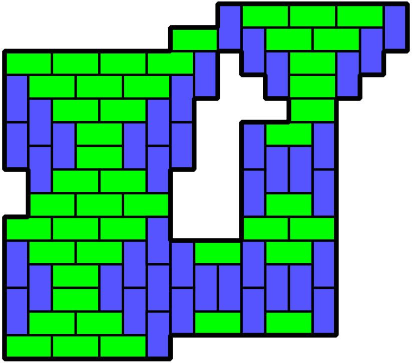

(Ruskey, 2009)

INPUT: A region, R, with n grid squares.
QUESTION: Can R be tatami covered with dominoes?

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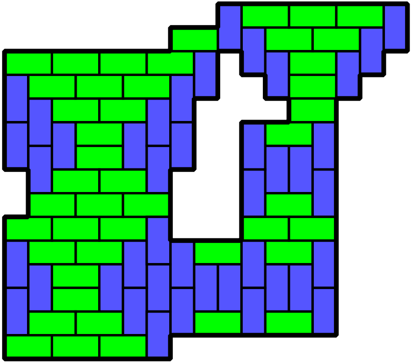

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INPUT: A region, R, with n grid squares.
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Is this NP-complete?

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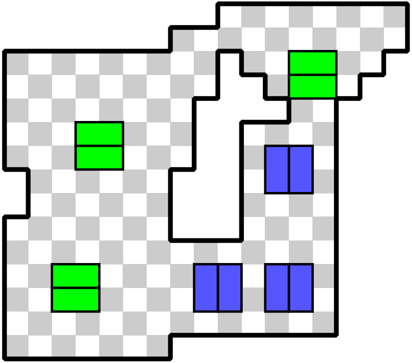

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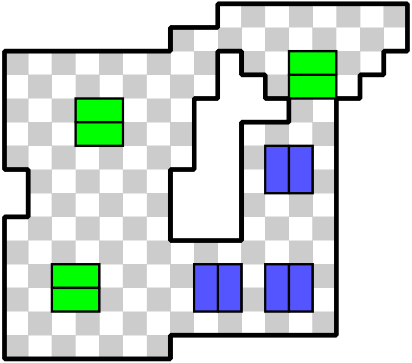

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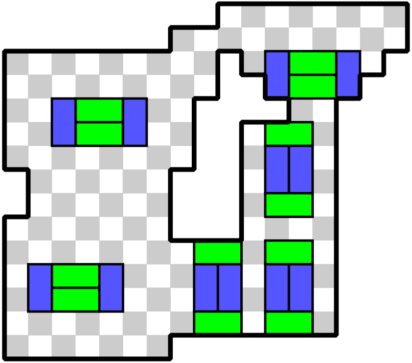

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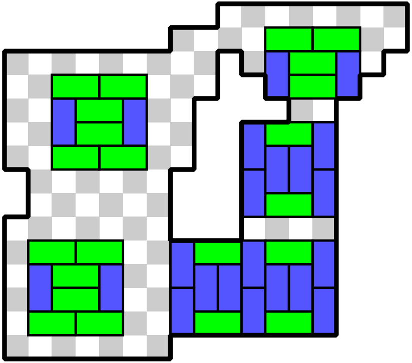

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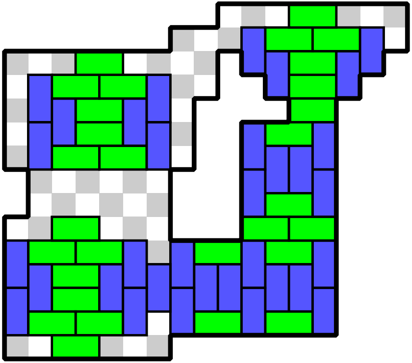

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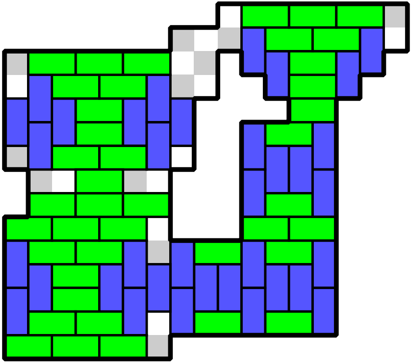

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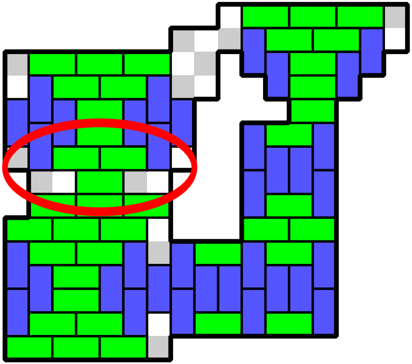

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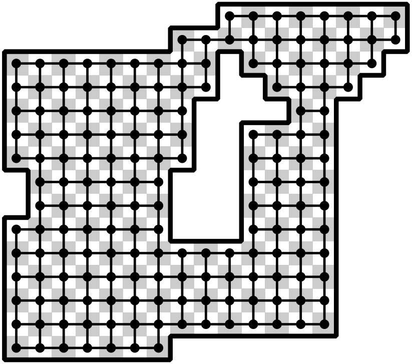
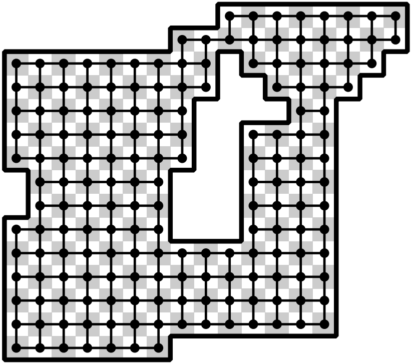
Domino ~~Tatami~~ Covering is polynomial



A domino covering is a perfect matching in the underlying graph.

Domino ~~Tatami~~ Covering is polynomial



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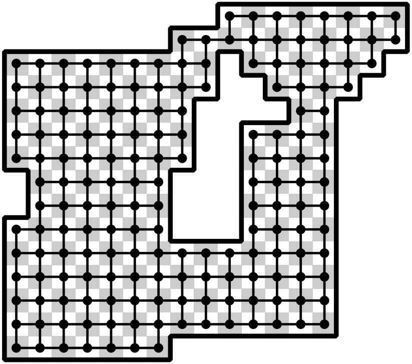

A domino covering is a perfect matching in the underlying graph.

INPUT: A region, R, with n grid squares.
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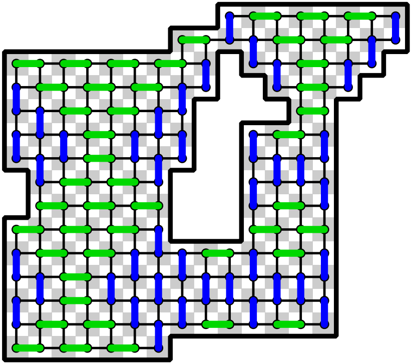
This can be answered in $O(n^2)$, since the underlying graph is bipartite.

Tatami coverings as matchings

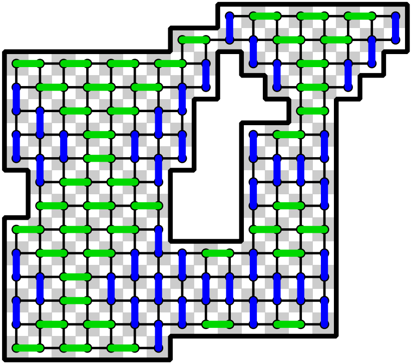


The tatami restriction is the additional constraint, that every 4-cycle contains a matched edge.

Theorem (Churchley, Huang, Zhu, 2011)

Given a graph G, it is NP-complete to decide whether it has a matching such that every 4-cycle contains a matched edge, even if G is planar.

Tatami coverings as matchings



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In Domino Tatami Covering, G is an induced subgraph of the infinite grid-graph, and the matching must be perfect.

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DTC is NP-complete

Domino Tatami Covering

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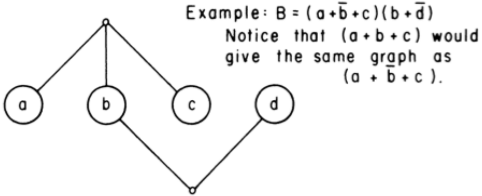
Theorem (E, Ruskey, 2013)

Domino Tatami Covering is NP-complete.

Planar 3SAT

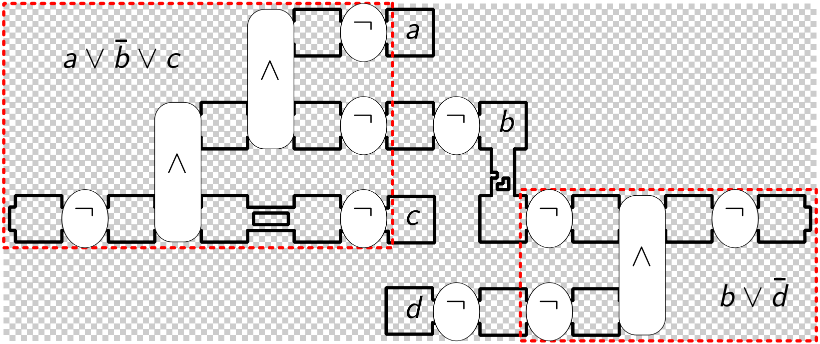
Let ϕ be a 3CNF formula, with variables U, and clauses C. Let $G = (U \cup C, E)$, where $\{u, c\} \in E$ iff one of the literals u or $\bar{u}$ is in the clause c. The formula is *planar* if there exists a planar embedding of G.

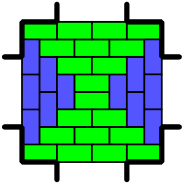


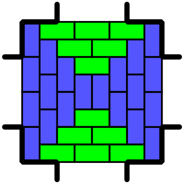
Planar 3SAT is NP-complete (Lichtenstein, 1982).

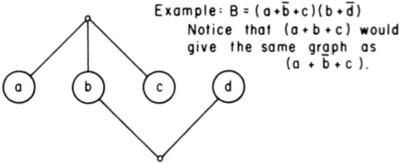
Reduction to Planar 3SAT

Working backwards from the answer...









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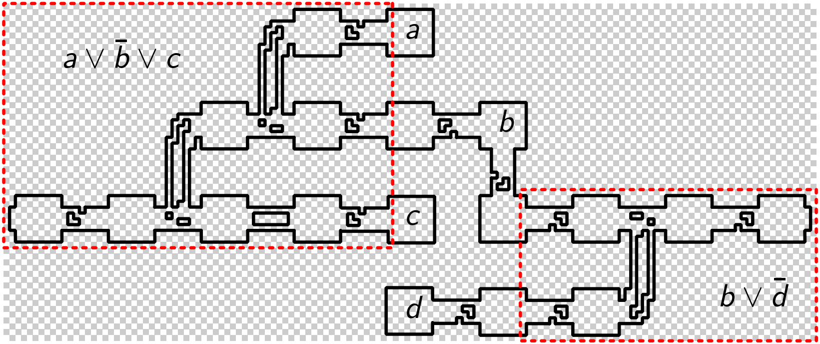

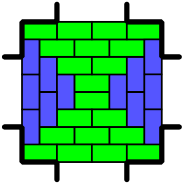

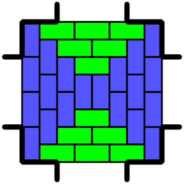

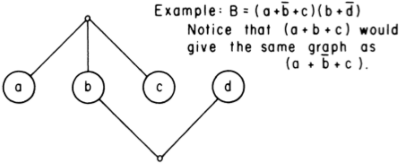

Reduction to Planar 3SAT

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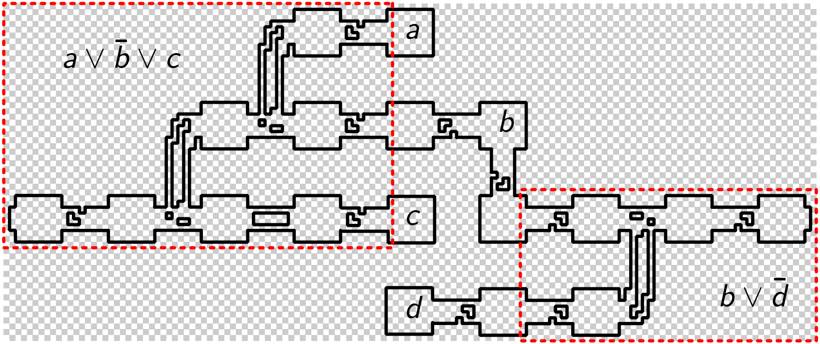
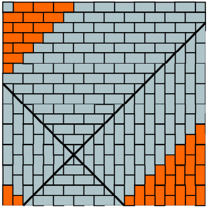
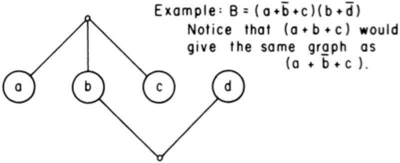
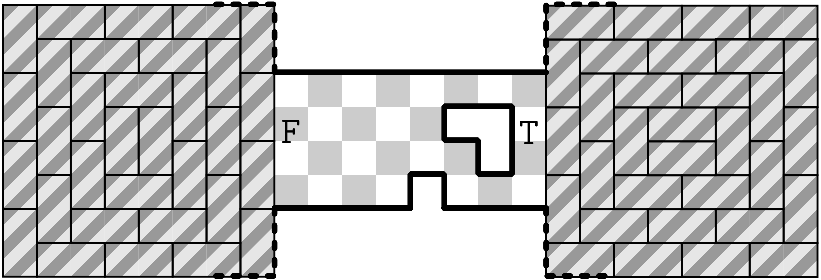






Verify the NOT gate



NOT gate covering can be completed with all "good" signals, but no "bad" signal.

"good"	"bad"
F $\longrightarrow$ T	T $\longrightarrow$ T
T $\longrightarrow$ F	F $\longrightarrow$ F

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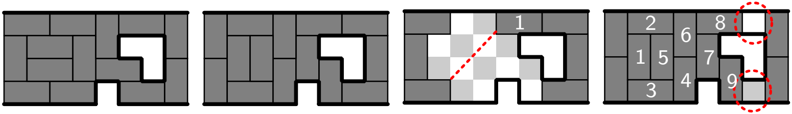

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"good"	"bad"
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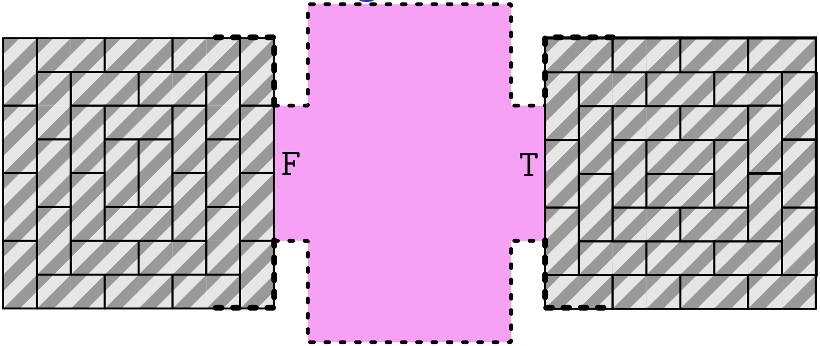
$T \longrightarrow F$	$F \longrightarrow F$

Search for a NOT gate



Search for sub-region, R, of the pink area. If R and the chessboards can be covered with all "good" signals, but no "bad" signal, we are done!

"good"	"bad"
$\mathtt{F} \longrightarrow \mathtt{T}$	$\mathtt{T} \longrightarrow \mathtt{T}$
$\mathtt{T} \longrightarrow \mathtt{F}$	$\mathtt{F} \longrightarrow \mathtt{F}$

SAT-solvers

- A SAT-solver is software that finds a satisfying assignment to a Boolean formula, or outputs UNSATISFIABLE. We used MiniSAT.
- Given an instance of DTC, the corresponding SAT instance has the edges of the underlying graph G, as variables. A satisfying assignment sets matched edges to TRUE and unmatched edges to FALSE.
- Three conditions must be enforced:
  1. TRUE edges are not incident.
  2. An edge at each vertex is TRUE.
  3. An edge of each 4-cycle is TRUE.

SAT-solvers

We can generate, test cover, and forbid regions with SAT-solvers.

```
4
12
CC#......#CC
CC#......#CC
CC#......#CC
CC#......#CC
2
.A........<>
.V........A.
.A........V.
.V........<>

<>........A.
.A........V.
.V........A.
<>........V.
```

```
2
<>........<>
.A........A.
.V........V.
<>........<>

.A........A.
.V........V.
.A........A.
.V........V.
```

Combine python scripts with the SAT-solver MiniSAT (fast, lightweight, pre-compiled for my system.)

Gadget Search

- request candidate region, R, from MiniSAT, satisfying “good” signals.
- MiniSAT to test each “bad” signal in R.
- if every test UNSATISFIABLE R is the answer!
- Else, “forbid” R in next iteration.

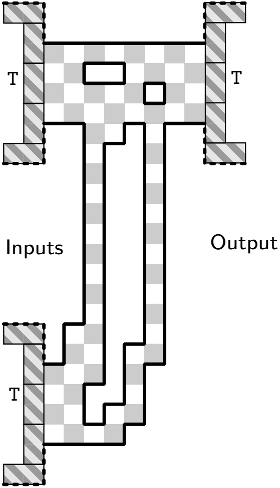
```python
numRegions = 0 #count the number of regions we have tried
prevR = []
while(True):
    numRegions += 1
    sp = subprocess.Popen(['./minisat',satinFilename,satoutFilename],stdout=subpr
ocess.PIPE)
    sp.wait()
    if(numRegions%100 == 0):
        print "number of regions checked", numRegions
    if(sp.returncode==10): #satisfied
        g = getSATAssignment(satoutFilename)
        R = g[:r*c] #the region output from last minisat of f
        if(prevR == R):
            quitError('error: two regions the same')
        if(numRegions%100 == 0):
#            print R
            displayRegion(R)
            print "good configurations"
            for k in range(C):
                displayTiling(g,k)
        prevR = R
        rClauses = '' #make clauses to enforce that region
        for _clause in R:
            rClauses = rClauses + str(_clause) + ' 0\n'
        badFlag = False
        for k in range(badC):
            #for each bad configuration, check if it can be completed
            #in the region R.
            badConfig = open(badsatinFilename,'w')
            badConfig.write(badCNFstring[k] + rClauses)
            badConfig.close()
            sp = subprocess.Popen(['./minisat',badsatinFilename,badsatoutFilename
],stdout=subprocess.PIPE)
            sp.wait()
            if(sp.returncode==10):
                badFlag = True
                if(numRegions%100==0):
                    print 'bad configuration'
                    displayTiling(getSATAssignment(badsatoutFilename),0)
                break
            elif(sp.returncode != 20):
                quitError('bad minisat returned bad code: ' + str(sp.returncode))
        if(badFlag == False):
            #we have found a good region!
            print "HORRAY", R
            sys.stdout.flush()
            sys.exit(0)
       #we are going to append a forbidden region to satinFilename
        f = open(satinFilename,'r+')
        #change the first line with the number of clauses
        f.seek(0,0)
        f.write('p cnf ' + str(nGoodVars) + ' ' + str(len(goodClauses)) + '\n')
        #make a clause from the forbidden region
        clause(map(neg,R))
        CNFstring = ''
        for lit in goodClauses[-1]:
            CNFstring = CNFstring + ' ' + str(lit)
        CNFstring = CNFstring + ' 0\n'
        #append this to the end of the file
        f.seek(0,2)
        f.write(CNFstring)
        f.close()
    elif(sp.returncode != 20):
        quitError('good minisat returned bad code: ' + str(sp.returncode))
    else:
        sys.stdout.write('There is no region that satisfies the input.')
        sys.stdout.flush()
        sys.exit(0)
```

Huge search space

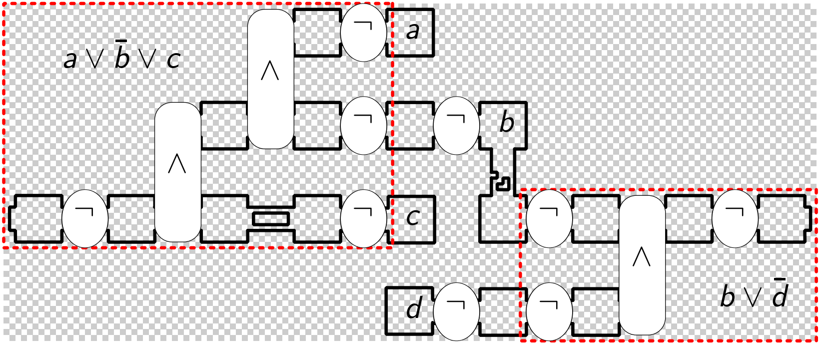
```
CC#....#CC
CC#....#CC
CC#....#CC
CC#..#.#CC
XXX.#..XXX
XXX..#.XXX
CC#.#..XXX
CC#....XXX
CC#....XXX
CC#....XXX
```

Require and forbid some grid squares (`#`, `X`) to be in R to reduce number of disconnected regions. Search a smaller area.

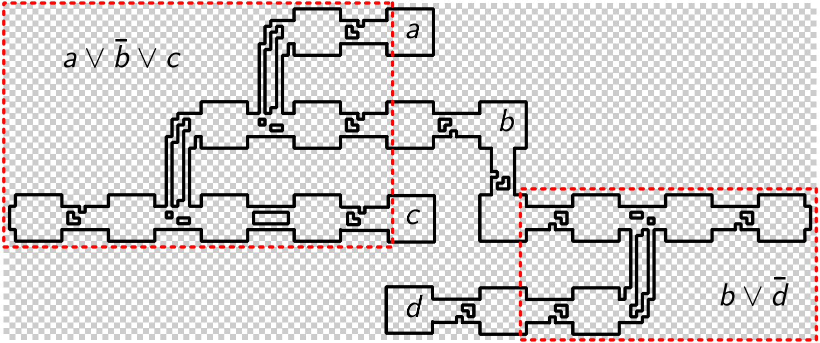
It worked!



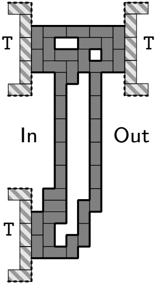
Recall the context



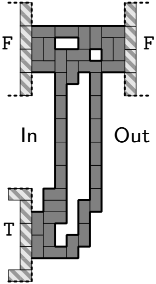
Recall the context



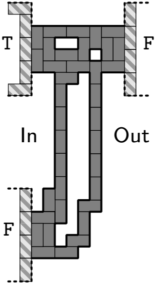
Verifiable by hand



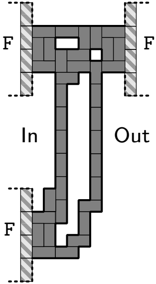
TT⟶T



TF⟶F

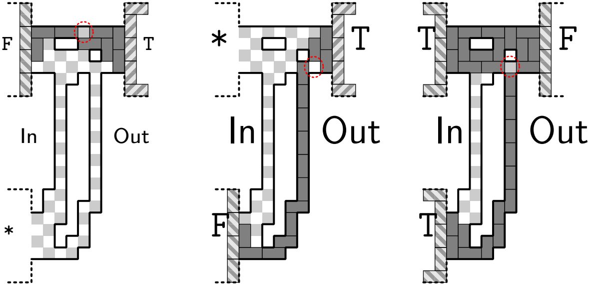


FT⟶F



FF⟶F

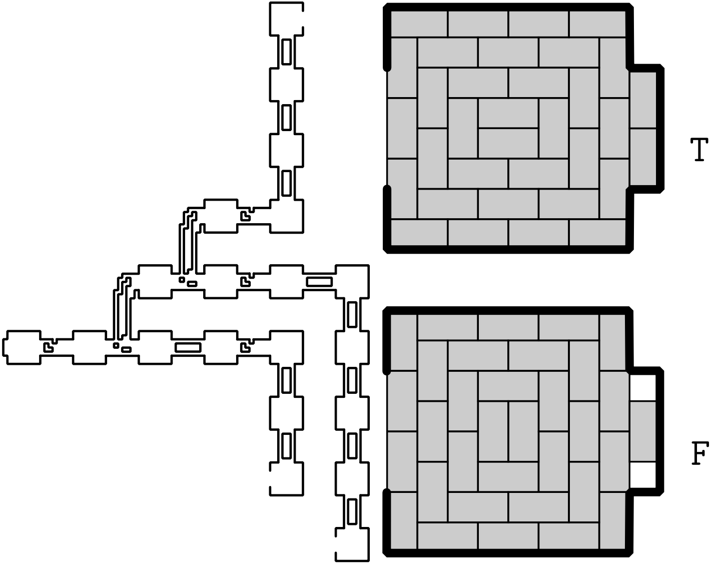
Verifiable by hand



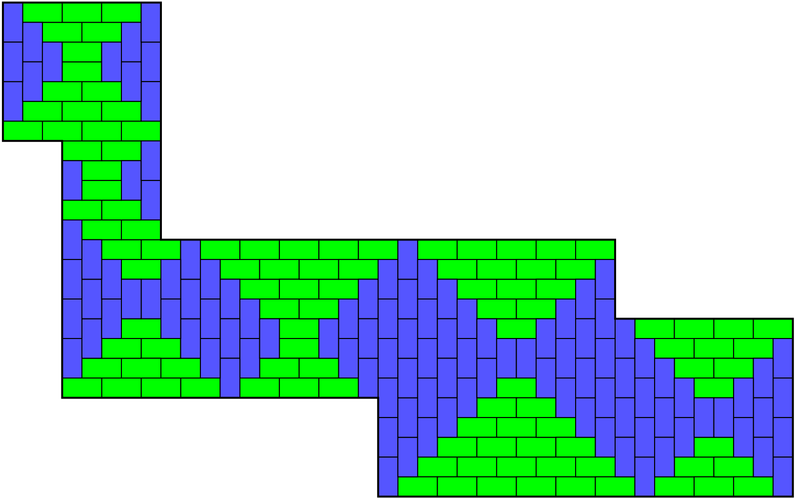
$*F \longrightarrow T$ $F* \longrightarrow T$ $TT \longrightarrow F$

Impossible AND gate coverings, where $*$ denotes F or T.

Testing a clause

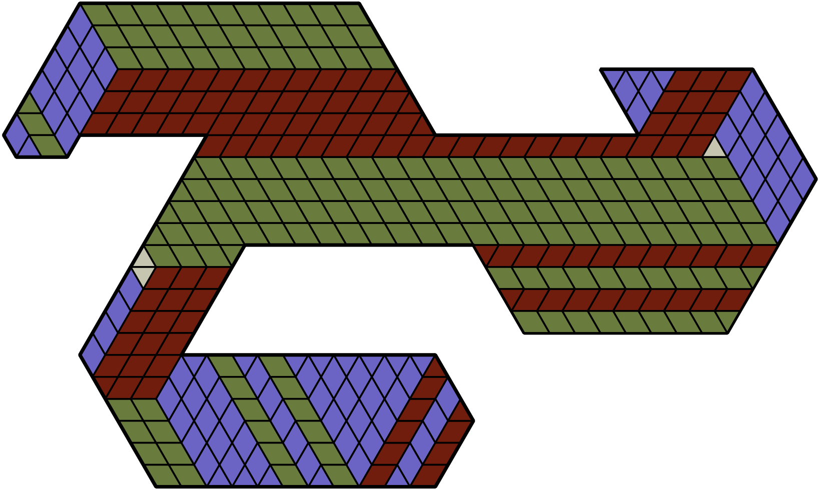


Simply Connected DTC

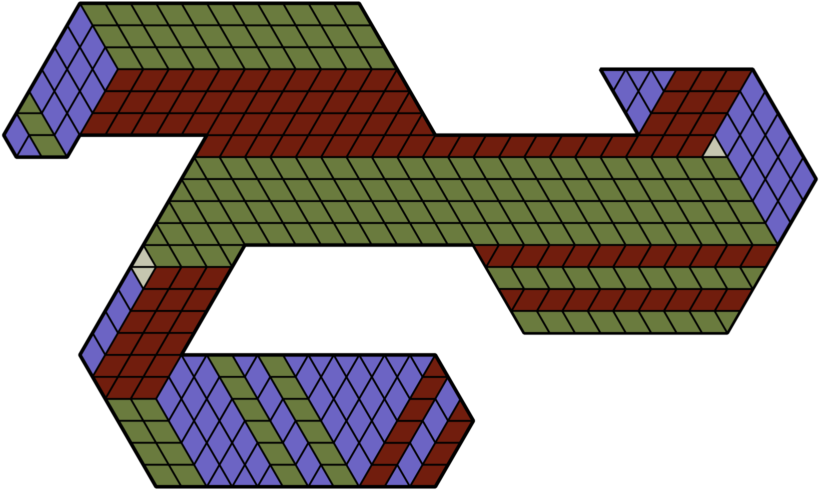


Is DTC NP-hard even if the region is simply connected?

Lozenge 5-Tatami Covering



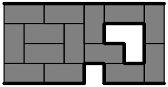
Lozenge 5-Tatami Covering



Is Lozenge 5-Tatami Covering NP-hard?

Domino +-Tatami Covering

What if we forbid tiles from meeting corner to corner? This was mildly advocated by Don Knuth, but it conflicts somewhat with the broader tatami structure.



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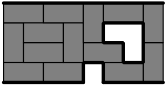
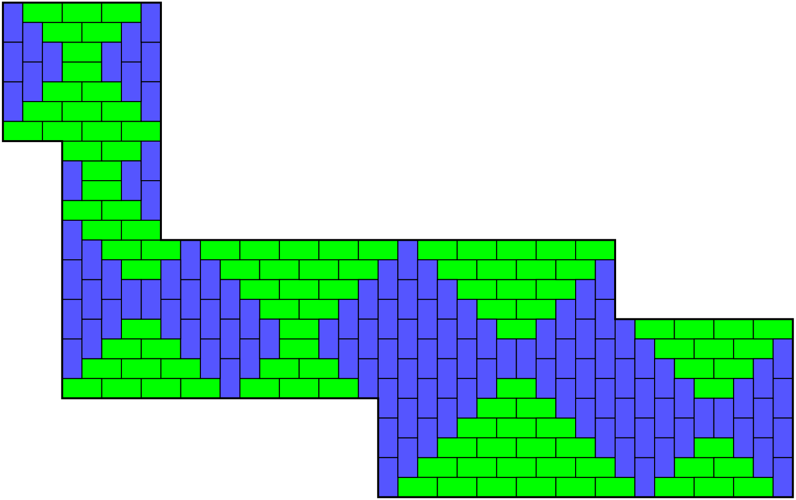
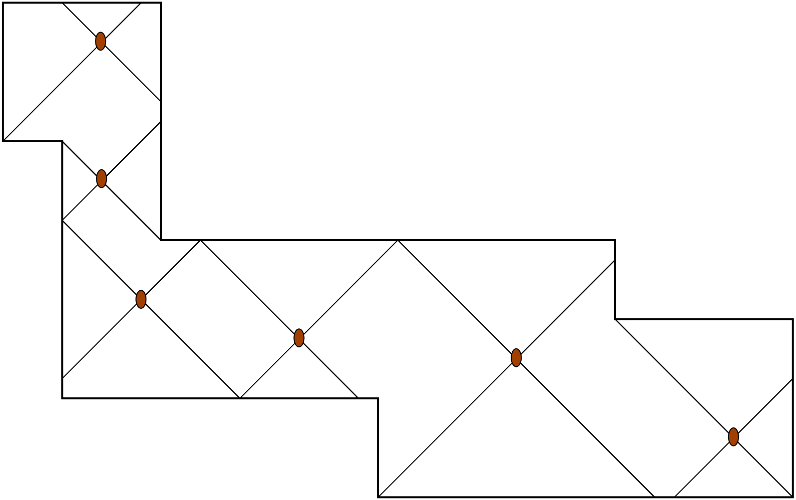


Is Domino +-Tatami Covering NP-hard?

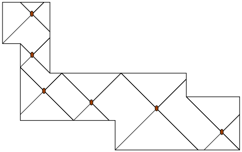
Water Strider Problem



Water Strider Problem



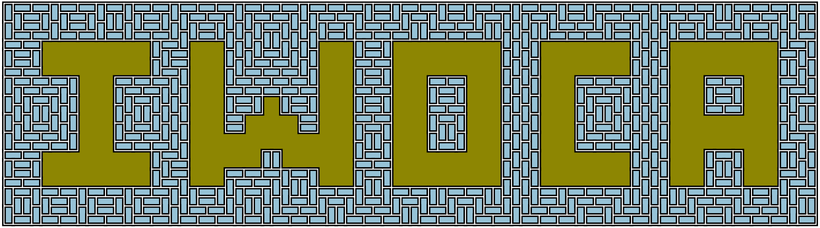
Water Strider Problem



INSTANCE: A rectilinear region, R, with n segments, and vertices in $\mathbb{R}^2$.

QUESTION: Is there a configuration of at most k water striders, such that no two water striders intersect, and no more water striders can be added?

Thank you



Thanks also to Bruce Kapron and Don Knuth. Part of this research was conducted at the 9th McGill-INRIA Workshop on Computational Geometry.