

- We make two observations initially. First, if $A(0) = 0$ then $A(m) = \sum_{1 \leq n \leq m} (A(n) - A(n-1))$ because all terms cancel each other except $A(m)$. Secondly, if $t(2k+1) = n+k$, then multiplying by 2 and subtracting $2k+1$ we obtain the middle equation below:

$$(2t-1)(2k+1) = 2t(2k+1) - (2k+1) = 2(n+k) - (2k+1) = 2n-1.$$

Thus, since any divisor of an odd number must be odd,

$$[2k+1 \mid n+k] = [2k+1 \mid 2n-1].$$

We can now proceed as follows:

$$\begin{aligned} A(m) &= \sum_{k=0}^{m-1} \phi(2k+1) \left\lfloor \frac{m+k}{2k+1} \right\rfloor \\ &= \sum_{n=1}^m (A(n) - A(n-1)) \quad \text{Note that } A(0) = 0. \\ &= \sum_{n=1}^m \left(\sum_{k=0}^{n-1} \phi(2k+1) \left\lfloor \frac{n+k}{2k+1} \right\rfloor - \sum_{k=0}^{n-2} \phi(2k+1) \left\lfloor \frac{n-1+k}{2k+1} \right\rfloor \right) \\ &= \sum_{n=1}^m \sum_{k=0}^{n-1} \phi(2k+1) \left(\left\lfloor \frac{n+k}{2k+1} \right\rfloor - \left\lfloor \frac{n-1+k}{2k+1} \right\rfloor \right) \\ &= \sum_{n=1}^m \sum_{k=0}^{n-1} \phi(2k+1) [2k+1 \mid n+k] \\ &= \sum_{n=1}^m \sum_{k=0}^{n-1} \phi(2k+1) [2k+1 \mid 2n-1] \\ &= \sum_{n=1}^m \sum_{\substack{k=0 \\ 2k+1 \mid 2n-1}} \phi(2k+1) \\ &= \sum_{n=1}^m (2n-1) \\ &= m^2. \end{aligned}$$

- Define $a_0 = 0$. Note that

$$\begin{aligned} a_n - a_{n-1} &= \sum_{k=1}^n b_k \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right) \\ &= \sum_{k=1}^n b_k [k \mid n] \\ &= \sum_{k \mid n} b_k. \end{aligned}$$

Now apply Möbius inversion to get

$$b_n = \sum_{k \mid n} \mu(n/k)(a_k - a_{k-1}).$$

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• > x := chrem([0, -1, -2, -3], [3*(2*7), 17*(11*13)*19, 23*31, 29*37]);  
y := chrem([0, -1, -2, -3], [3*(2*11), 23*(7*13)*29, 17*37, 19*31]);  
x := 911942452134  
y := 914436662502  
> for i from 0 to 3 do seq(igcd(x+i, y+j), j = 0 .. 3) end do;  
6, 7, 2, 3  
11, 13, 17, 95  
2, 23, 8, 31  
3, 29, 37, 3
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