CSC 482/582 Summer 2012 Quiz #1 NAME: I.M. Solution

Instructions: Closed book and notes. Answer all questions.

1. [5 marks]

   (a) Simplify: $\lfloor n/2 \rfloor + \lceil n/2 \rceil = \underline{n}$.

   (b) Simplify: $-\lfloor -x \rfloor = \underline{\lceil x \rceil}$?

2. [5 marks] Recall that $J_2(n)$ is the Josephus function studied in Chapter 1.

   (a) What is $J_2(32+16+4+2)$? ANSWER: $(101101)_2 = 45$

   (b) Characterize those values of n for which $J_2(n) = n$.

   ANSWER: $n = 2^m - 1$ for some $m > 0$.

3. [5 marks] Below we are interchanging the order of three sums.

$$\sum_{1 \le j \le n} \sum_{1 \le k \le j} \sum_{1 \le l \le k} f(j,k,l) = \sum_{?_1 \le l \le ?_2} \sum_{?_3 \le k \le ?_4} \sum_{?_5 \le j \le ?_6} f(j,k,l)$$

   What are $?_1, ?_2, ?_3, ?_4, ?_5, ?_6$. Justify your answer by writing out some equalities involving the $[\![P]\!]$ notation.

   ANSWER: Let X_n be the sum above on the left.

$$\begin{aligned} X_n &= \sum_{j,k,l} f(j,k,l)[\![1 \le j \le n]\!][\![1 \le k \le j]\!][\![1 \le l \le k]\!] \\ &= \sum_{j,k,l} f(j,k,l)[\![1 \le l \le k \le j \le n]\!] \\ &= \sum_{j,k,l} f(j,k,l)[\![1 \le l \le n]\!][\![l \le k \le n]\!][\![k \le j \le n]\!] \\ &= \sum_{1 \le l \le n} \sum_{l \le k \le n} \sum_{k \le j \le n} f(j,k,l) \end{aligned}$$

4. [10 marks]

(a) What is $\Delta x^{\underline{3}}$? ANSWER: $3x^{\underline{2}}$.

(b) What function v, if any, satisfies $\Delta v(x) = 5^x$? ANSWER: $5^x/4$.

(c) Below is equation (2.55) from the book.

$$\sum u\ \Delta v = uv - \sum Ev\ \Delta u.$$

Use this once to get a "simpler" (it will still involve a summation) expression for

$$\sum_{k=1}^{n} k^{\underline{3}} 5^k = \left.\frac{x^{\underline{3}} 5^x}{4}\right|_1^{n+1} - \frac{15}{4} \sum_{k=1}^{n} k^{\underline{2}} 5^k$$

5. [5 marks] Solve the recurrence relation $D(0) = 1$ and for $n > 0$,

$$D(n) = \frac{n+2}{n} D(n-1) + 1.$$

You can use whatever method you want. I suggest that you compute a few small values of $D(n)$ first.

ANSWER: An examination of small values seems to indicate that $D(n) = (n+1)^2$. This is indeed the case and it is easy to verify by induction.

As an illustration, let's try the multiplicative factor approach; we then multiply through by $1/((n+1)(n+2))$ to obtain

$$\frac{D(n)}{(n+1)(n+2)} = \frac{D(n-1)}{n(n+1)} + \frac{1}{(n+1)(n+2)}.$$

This gives us the sum

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n} k^{\underline{-2}} = \left.\frac{k^{\underline{-1}}}{-1}\right|_0^{n+1} = 1 - \frac{1}{n+2}$$

by the finite calculus. Multiplying through by $(n+1)(n+2)$ gives $(n+1)^2$, as suspected.