CSC 482/582 Summer 2012 Quiz #4   NAME: _____

Instructions: Closed book and notes. Answer all questions.

1. [5 marks] What fraction is

$$\binom{1/3}{3}?$$

   ANSWER:
   $$\frac{(1/3)(1/3-1)(1/3-2)}{3!} = \frac{(1/3)(-2/3)(-5/3)}{6} = \frac{5}{81}.$$

2. [5 marks] How many sets A and B are there such that $A \subseteq B \subseteq \{1, 2, \ldots, n\}$ where $|A| = k$ and $|B| = m$?

   ANSWER: Either of the following:
   $$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

3. [5 marks] What is the generating function of the sequence defined by $[\![0 \leq k]\!]$? I.e., the sequence is $\langle 1, 1, 1, 1, 1, 1, \ldots \rangle$. How about the sequence defined by $[\![100 \leq k \leq 999]\!]$? Your answer should not have any summations or $\cdots$.

   ANSWER: Let $a_k = [\![100 \leq k \leq 999]\!]$. Notice that if $b_{100} = 1$, $b_{1000} = -1$, and otherwise $b_k = 0$, then $a_k = b_0 + b_1 + \cdots + b_k$. Since the generating function of the b_k is $z^{100} - z^{1000}$, the generating function of the a_k is
   $$\sum_{k \geq 0} a_k z^k = \frac{z^{100} - z^{1000}}{1 - z}.$$

   Of course, there are other ways to derive this answer.

4. [10 marks] Recall that the Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ if $n > 1$. Prove by induction on n that the equation below is true. HINT: Use the Pascal triangle recurrence relation.

$$F_{n+1} = \sum_{k \geq 0} \binom{n-k}{k}.$$

ANSWER: For this problem, we will assume that $\binom{m}{k} = 0$ if $m < 0$ (otherwise we must specify an upper limit on the sum above). The lower limit on the sum is not necessary since $\binom{n-k}{k} = 0$ if $k < 0$. The proof is by induction on n.

$$\sum_k \binom{n-k}{k} = \sum_k \left(\binom{n-k-1}{k} + \binom{n-k-1}{k-1} \right)$$

$$= \sum_k \binom{n-k-1}{k} + \sum_k \binom{n-k-1}{k-1}$$

$$= \sum_k \binom{(n-1)-k}{k} + \sum_k \binom{(n-2)-(k-1)}{k-1}$$

$$= F_n + \sum_k \binom{(n-2)-k}{k}$$

$$= F_n + F_{n-1}.$$

For the base cases we have

$$F_0 = \sum_{k \geq 0} \binom{-1-k}{k} = 0, \qquad \text{and}$$

$$F_1 = \sum_{k \geq 0} \binom{0-k}{k} = \binom{0}{0} = 1.$$