CSC 320: Proof of the Day

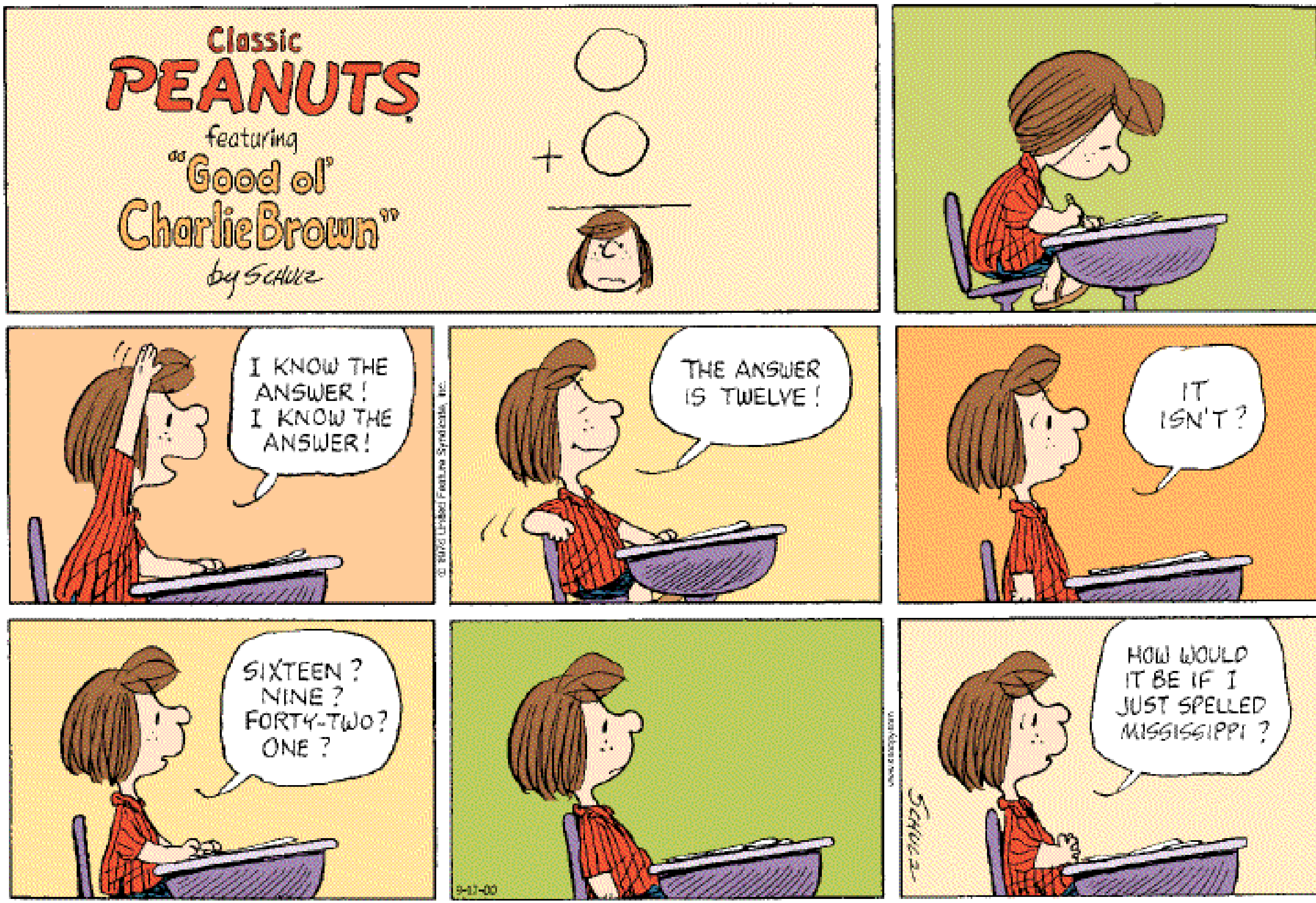
Definition: for functions f and g which map the natural numbers to real values, the function f(n) is in O(g(n)) if there exists constants $c > 0$ and $n_o \geq 0$ such that for all

$n \geq n_0$, $f(n) \leq c \cdot g(n)$.

Use this definition to prove that

$3 + 4n + 5n^2 + 2n^3$ is in $O(n^3)$.

You will learn a lot more if you try the problems and get them wrong than if you do not try. Also, it helps me know where the class is in terms of understanding.



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Announcements

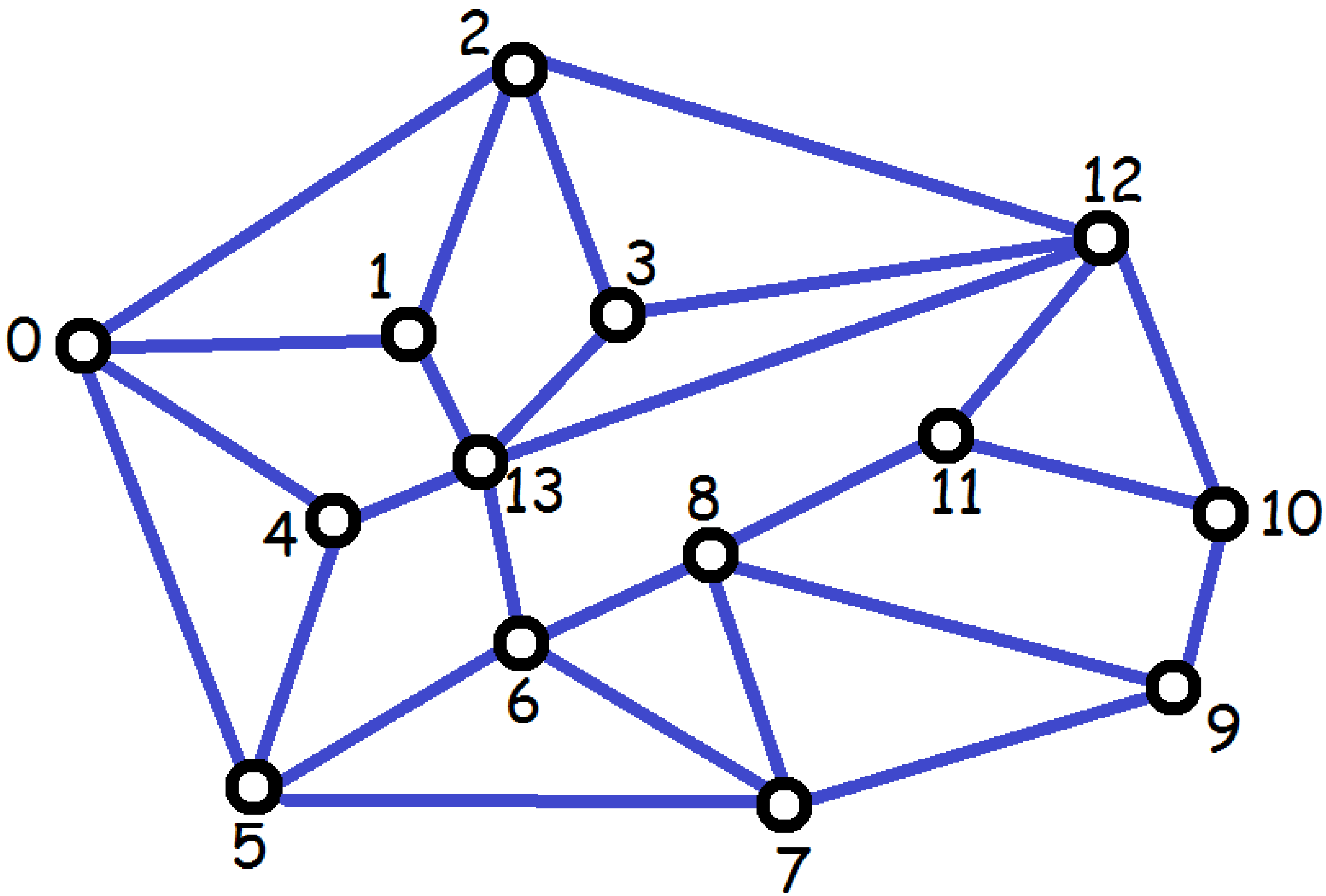
Assignment #1 and Tutorial #1 are posted.
Tutorials start next week.

Please move closer to the front of the class if you have any problems hearing or seeing the board or the slides.

If you had problems with the induction question, you are not alone, most students did not have nice proofs. If you understand induction instead of memorizing a template it will be harder to forget how to do it.

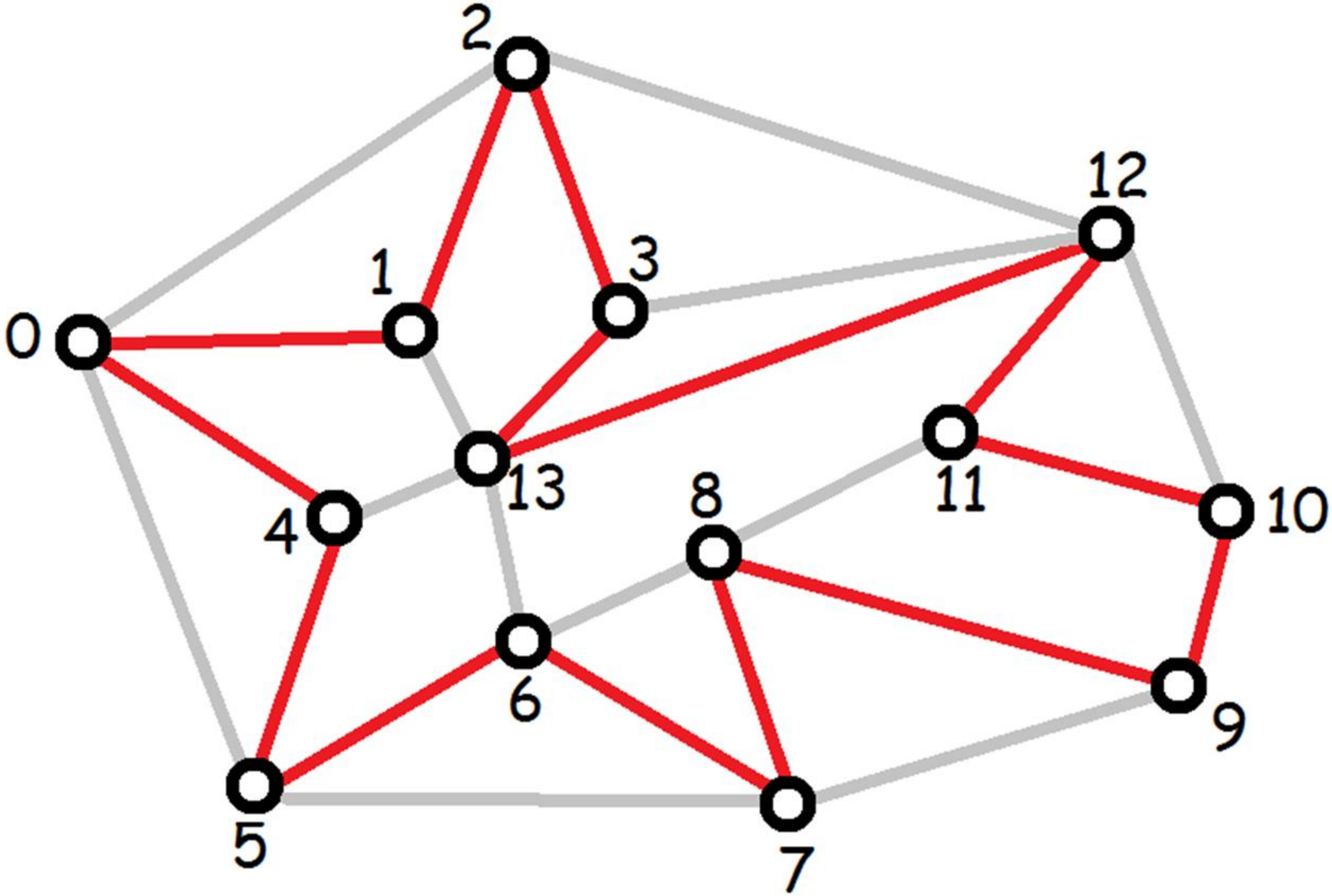
Hamilton Cycle:

A cycle which includes all the vertices of a graph.



Hamilton Cycle:

A cycle which includes all the vertices of a graph.

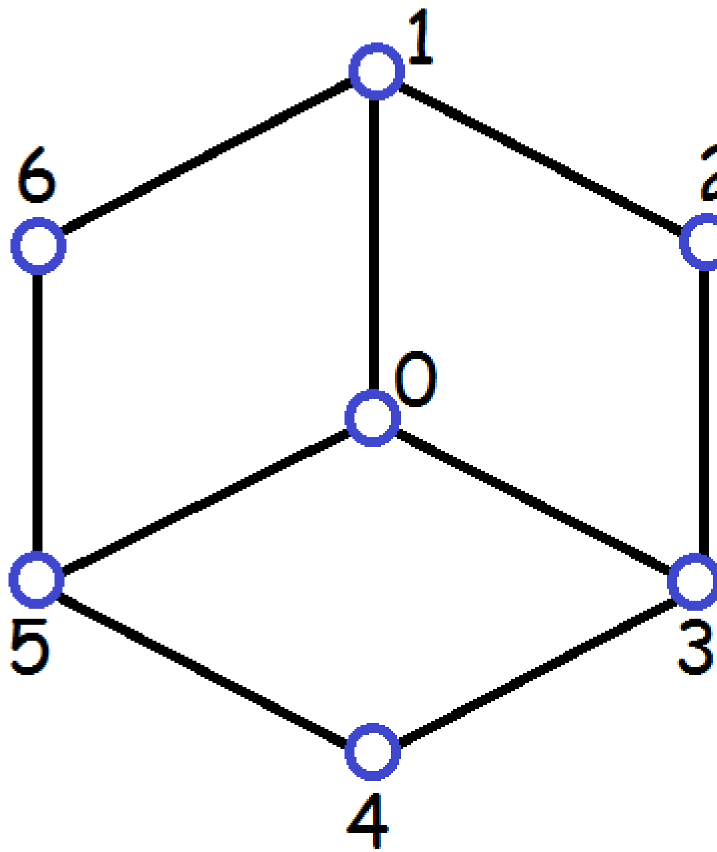


Certificate :0, 1, 2, 3, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4

Hamilton Cycle:

A cycle which includes all the vertices of a graph.

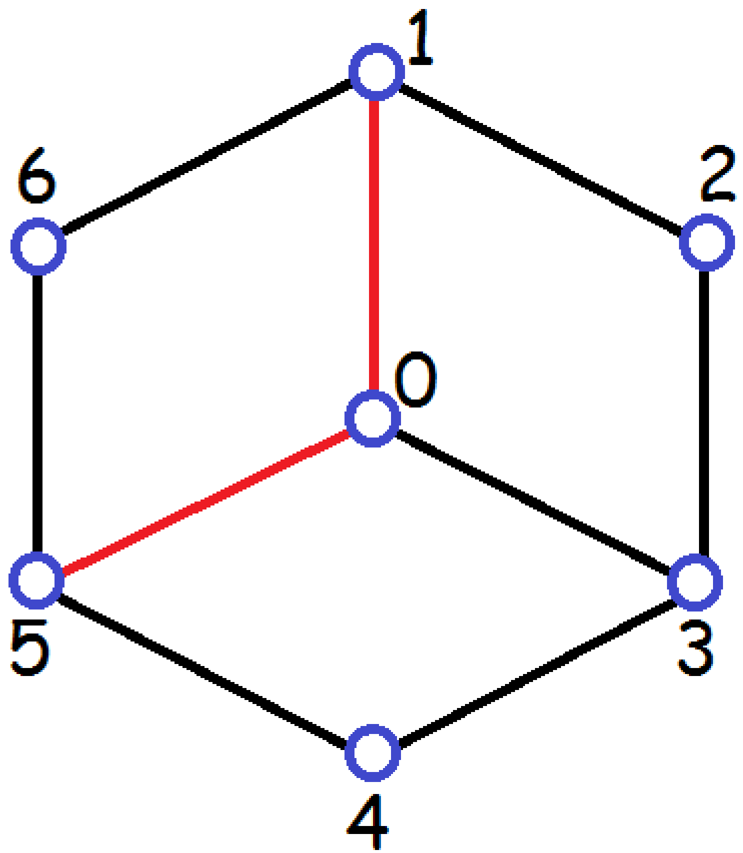
This graph has no Hamilton cycles:



Hamilton Cycle:

A cycle which includes all the vertices of a graph.

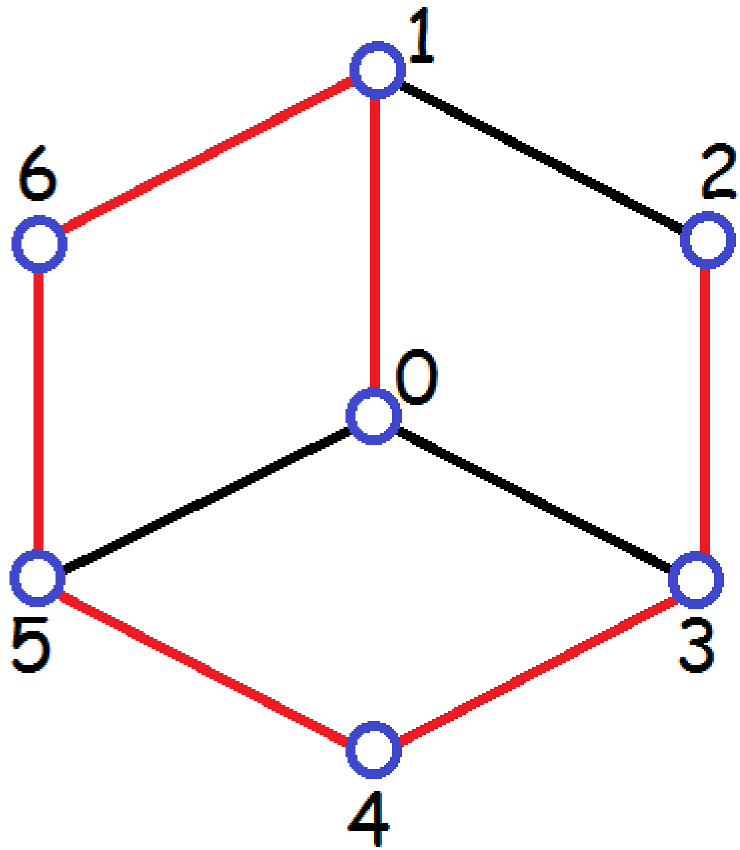
This graph has no Hamilton cycles:



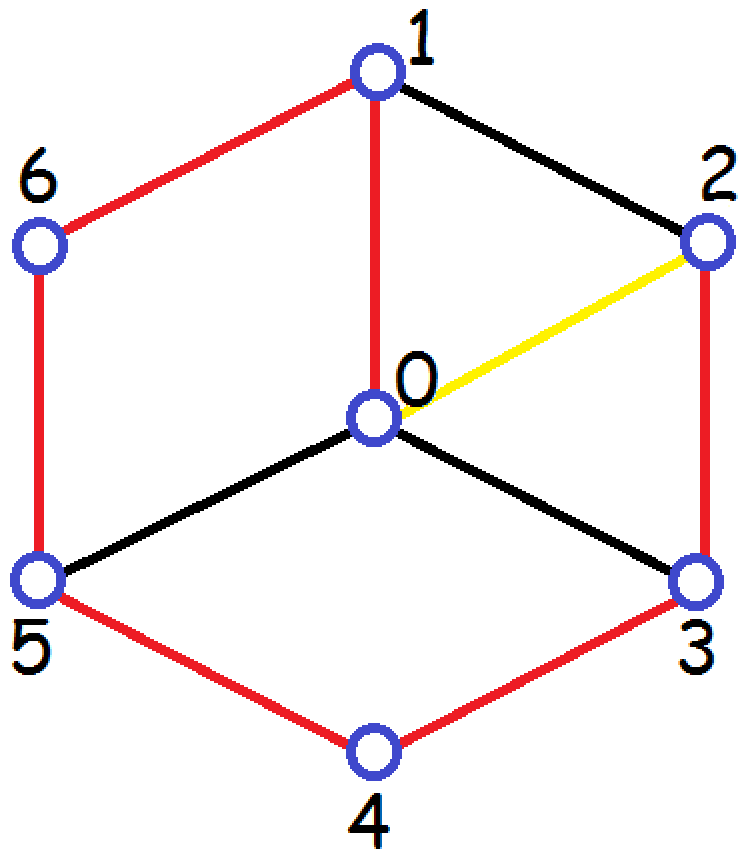
Hamilton Path:

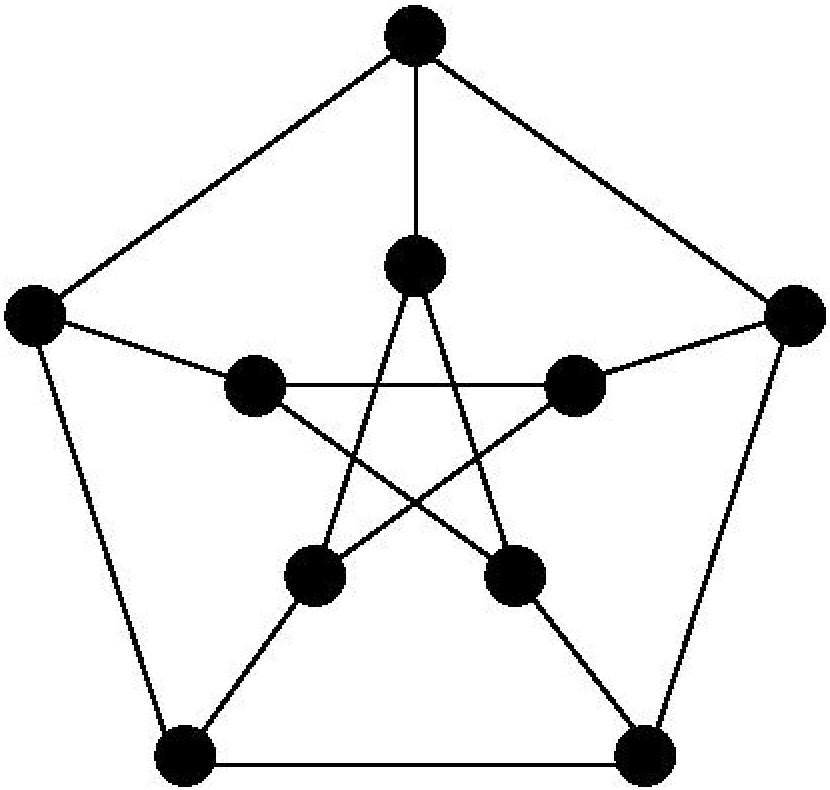
A path which includes all the vertices of a graph.

This graph has a Hamilton path:



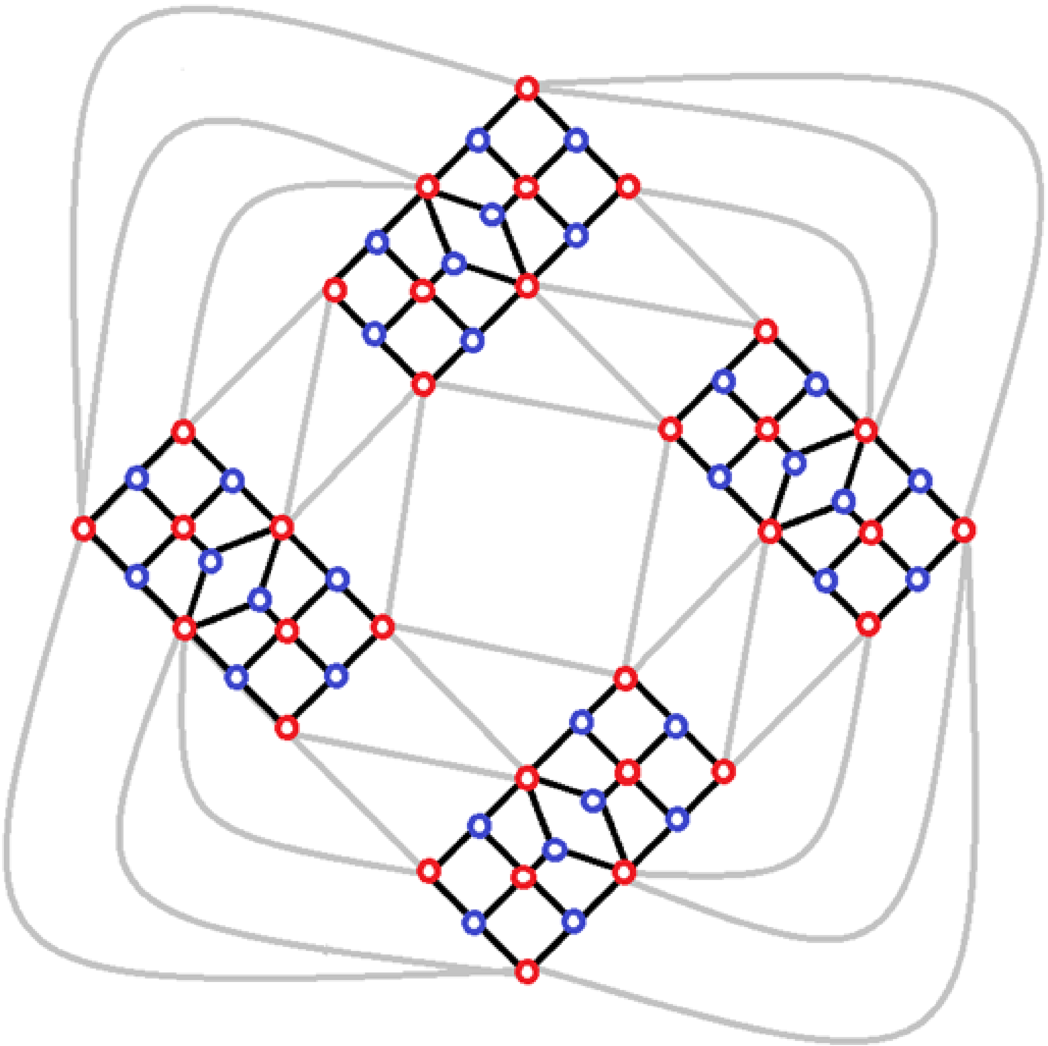
This graph has a Hamilton cycle:



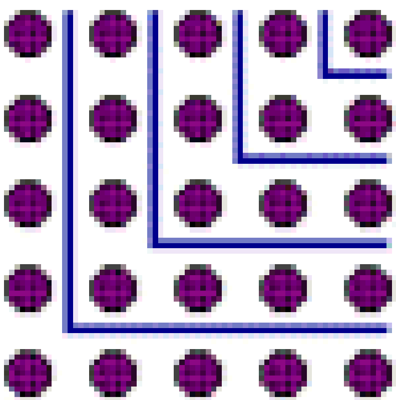


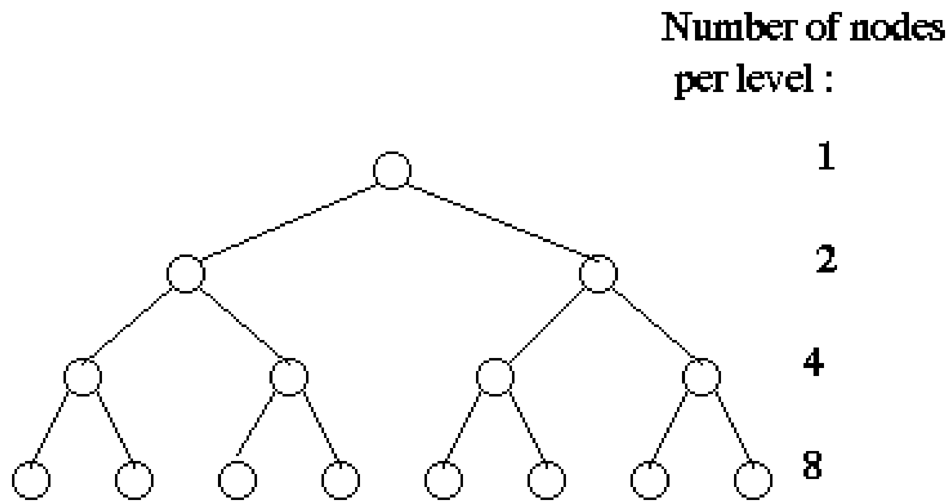
The Petersen graph has no Hamilton cycles.

This graph has no Hamilton cycles:



Review of Induction





$$n^2 = 1 + 3 + 5 + \ldots + (2n - 1)$$

Induction

Induction is one of the simplest proof tactics.

A deep understanding of what it means to prove something mathematically, and of induction is a good foundation for when we do other more sophisticated proofs later in the class.

http://soffl.deviantart.com/art/Larry-the-Pumping-Lemming-163624515

Natural Numbers

$\mathbb{N} = \{ 0, 1, 2, 3, 4, \ldots \}$

Inductive Definition:

[Basis] 0 is in the set $\mathbb{N}$

[Inductive step]:

If k is in $\mathbb{N}$

then k+1 is in $\mathbb{N}$

Last class:

Prove by induction that the number of binary strings of length k is 2^k.

For example:

The binary strings of length 3 are:

000, 001, 010, 011,
100, 101, 110, 111

and there is 8 of them, and $8 = 2^3$.

Why do we write proofs?

Why do we write proofs?

To convince others that a particular mathematical statement is true. The arguments need to be convincing and easy to read.

Prove by induction that the number of binary strings of length k is 2^k.

My base case: k=0.
There is only one binary string of length 0, the empty string (which in CSC 320 we will denote by ε).

The formula states that the number of strings of length 0 is $2^k = 2^0 = 1$ and hence, the formula gives the correct value when k=0.

How do we know if we should include more cases in the base case or not?

For elegance: do not include unnecessary cases.

Assume that the number of binary strings of length k is 2^k. This mathematical statement is our induction hypothesis.

It is a good idea to write down what we are trying to prove next so that you will know you have arrived at the goal when you get there.

We want to prove that the number of binary strings of length k+1 is 2^{k+1}.

DIGRESSION:

How do we go from length k to length k+1?

It helps to start in this case with an inductive definition of a binary string.

Concatenation of strings x and y denoted $x \cdot y$ or simply xy means write down x followed by y.

Inductive definition of a binary string:

[Base Case] The empty string ε is a binary string.

[Induction step]

If w is a binary string, then the two strings

w 0

and

w 1

are also binary strings.

Continuing on with the proof:

Each binary string of length k+1 has a prefix of length k that is an arbitrary binary string of length k and this prefix is followed by one more symbol which is 0 or 1.

Recall our induction hypothesis: The number of binary strings of length k is 2^k.

By induction, the number of possible prefixes of length k is 2^k.

The number of choices for the last symbol is 2. Therefore, the total number of binary strings of length k+1 is equal to $2^k * 2 = 2^{k+1}$ as required.

Common problem in solutions submitted:

Using only algebra but no words or connections to strings. You will not get any marks for proofs like this in CSC 320.

Many students apply the induction hypothesis without explaining to the reader what you were doing. You will lose marks on the assignment if you do not explain where you are applying the induction hypothesis.

A proof is intended for someone to read. It will be easier for someone to understand and believe in your proof if you explain what you are doing algebraically at every step.

Your proof will be more elegant if you don't change the variable names (for this problem, stick to k instead of switching to n).

Some students made up some mathematical notation without defining what they meant by it.
Then the notation was used having inconsistent types.
For example:

$S(k) = 2^k$ This has S(k) set to be an integer value.

Then later they might write:
$S(k) \Rightarrow S(k+1)$.
In this case I would assume S(k) is a mathematical statement maybe:
S(k): "The number of binary strings of length k is 2^k."

You can catch a lot of errors in your CSC 320 writing by doing type checking and making sure when you say two things are the same their types match.

Prove by induction that the number of binary strings of length k is 2^k.

My base case: k=0.

Then I write in the induction step:

Assume that the number of binary strings of length k is 2^k for some $k \geq 1$.

We want to prove that the number of binary strings of length k+1 is 2^{k+1}.

What is wrong with this?

Induction:

I want you to:

1. Understand why it works as a proof technique.
2. Write proofs that explain clearly what you are doing at every step (except for very simple algebra). Be sure to mention where it is that you apply the induction hypothesis. Everything you write should be mathematically valid.
3. Be able to use it on novel applications (requires understanding).
4. If you try to prove a hypothesis that is not correct, I want you to indicate where and why the induction proof fails. You will get zero marks for "proofs" for incorrect statements.
5. Elegance is good (e.g. don't put more in the base case than you really need).