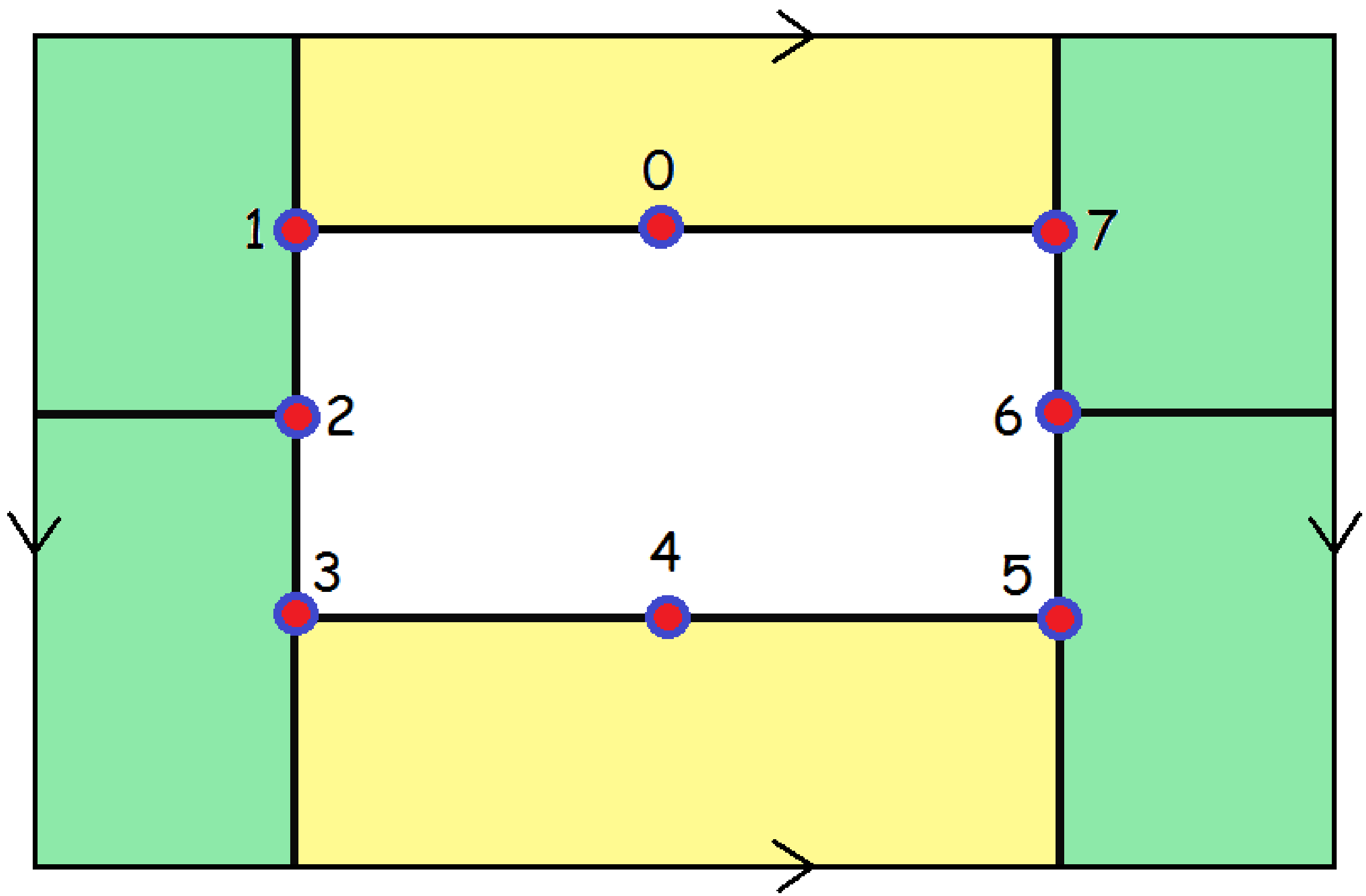
1. Walk the faces of this rotation system.
2. Is it on the plane or torus? Hint: on the plane, f= m-n+2, and on the torus, f= m-n.
3. Draw a picture of the embedding.
4. Is the embedding chiral or not?

0: 1 5 3
1: 0 6 2
2: 1 7 3
3: 0 4 2
4: 3 7 5
5: 0 4 6
6: 1 5 7
7: 2 6 4

First walk the faces of this rotation system then use that information to draw a picture of the embedding.
Recall: $g = (2 - n + m - f)/2$.

0: 1 7
1: 0 2 3
2: 1 3 6
3: 1 2 4
4: 3 5
5: 4 6 7
6: 2 5 7
7: 0 5 6

0: 1 7
1: 0 2 3
2: 1 3 6
3: 1 2 4
4: 3 5
5: 4 6 7
6: 2 5 7
7: 0 5 6



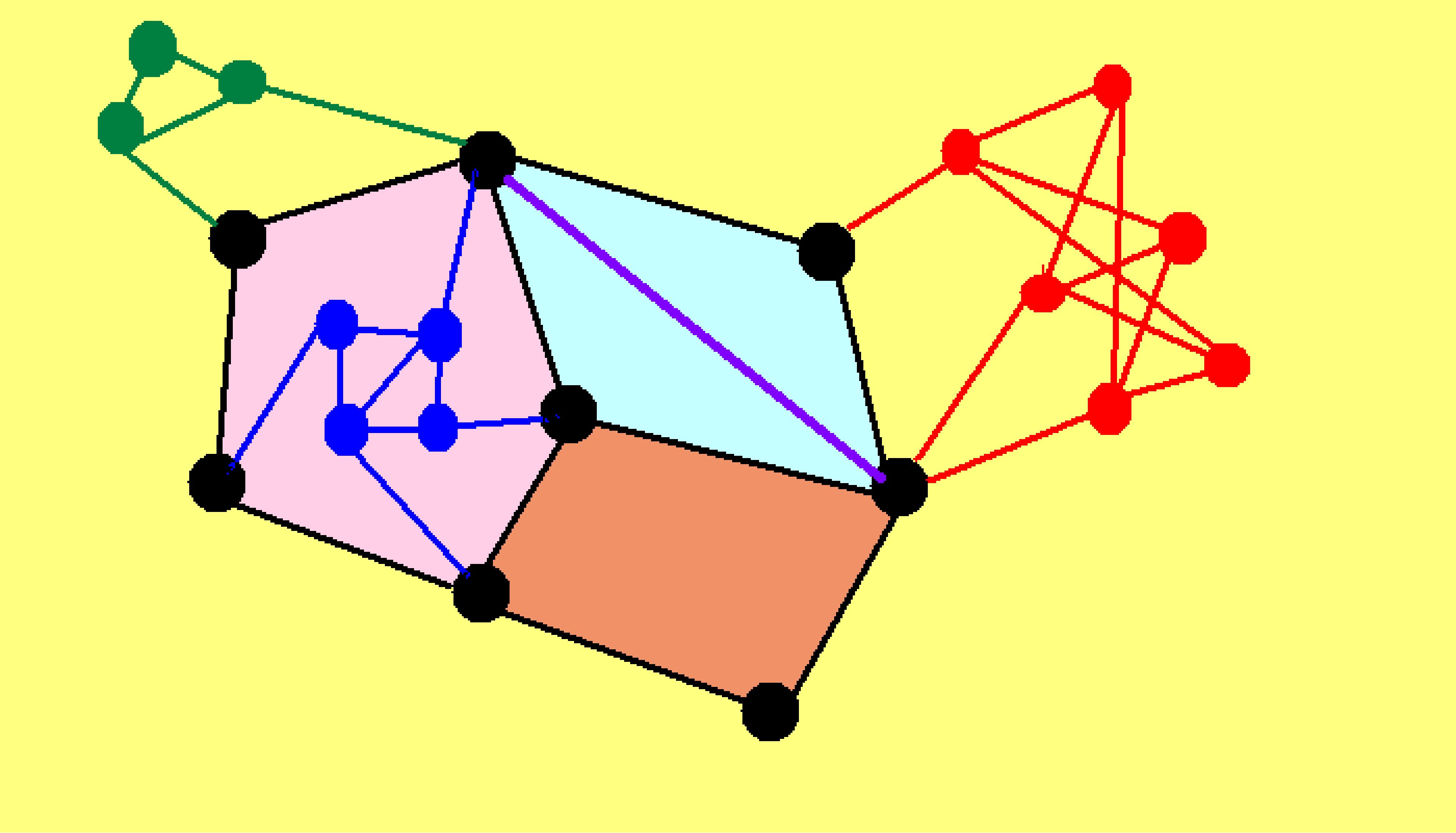
How can we find a planar embedding of each 2-connected component of a graph?
One simple solution: Algorithm by Demoucron, Malgrange and Pertuiset.

```
@ARTICLE{genus:DMP,
  AUTHOR = {G. Demoucron and Y. Malgrange
                and R. Pertuiset},
  TITLE = {Graphes Planaires},
  JOURNAL = {Rev. Fran\c{c}aise Recherche
                 Op\'{e}rationnelle},
  YEAR = {1964},
  VOLUME = {8},
  PAGES = {33--47}   }
```

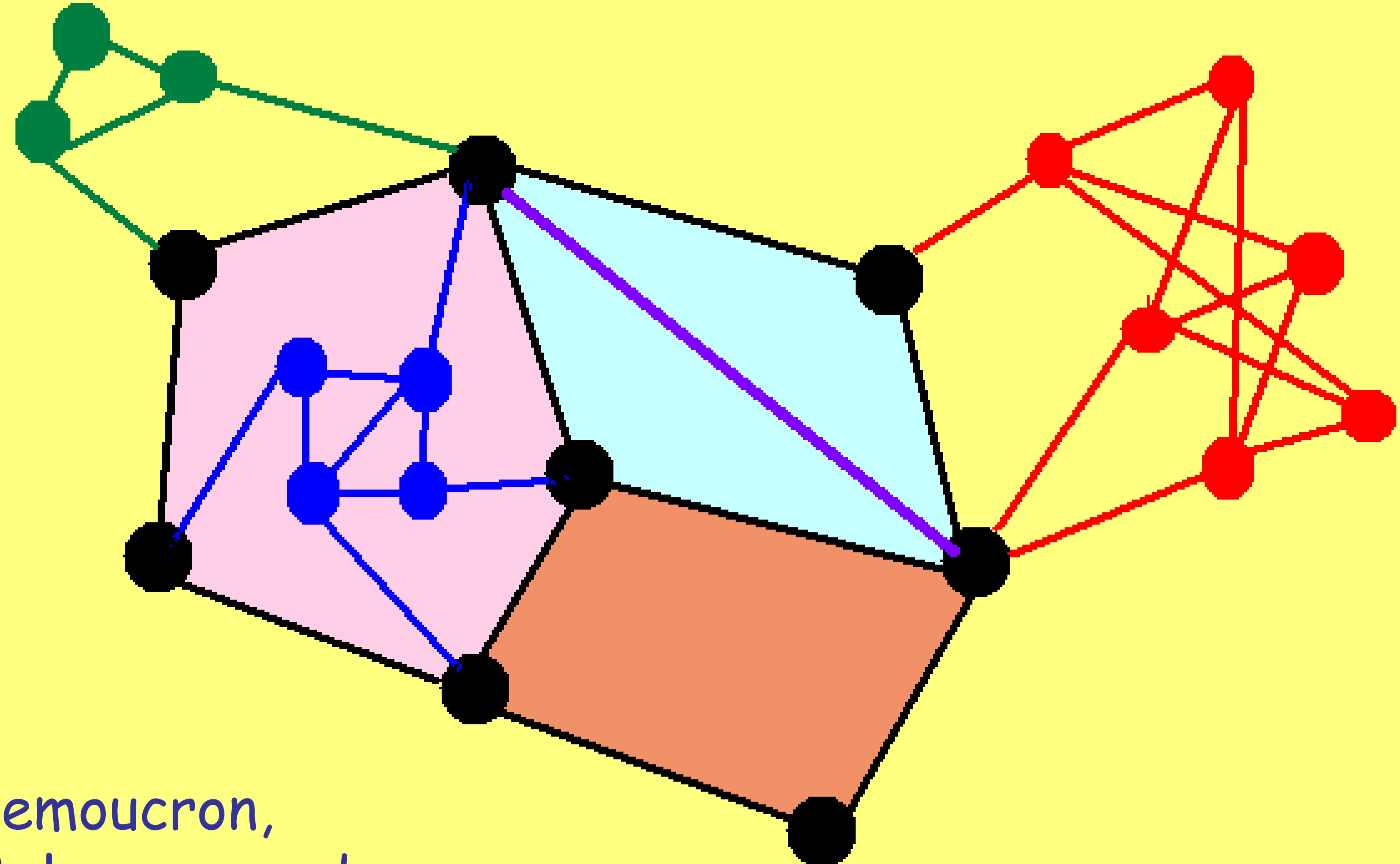
A **bridge** with respect to a subgraph H of a graph G is either:

1. An edge e=(u, v) which is not in H but both u and v are in H.
2. A connected component C of G-H plus any edges that are incident to one vertex in C and one vertex in H plus the endpoints of these edges.

How can you find the bridges with respect to a cut vertex v?

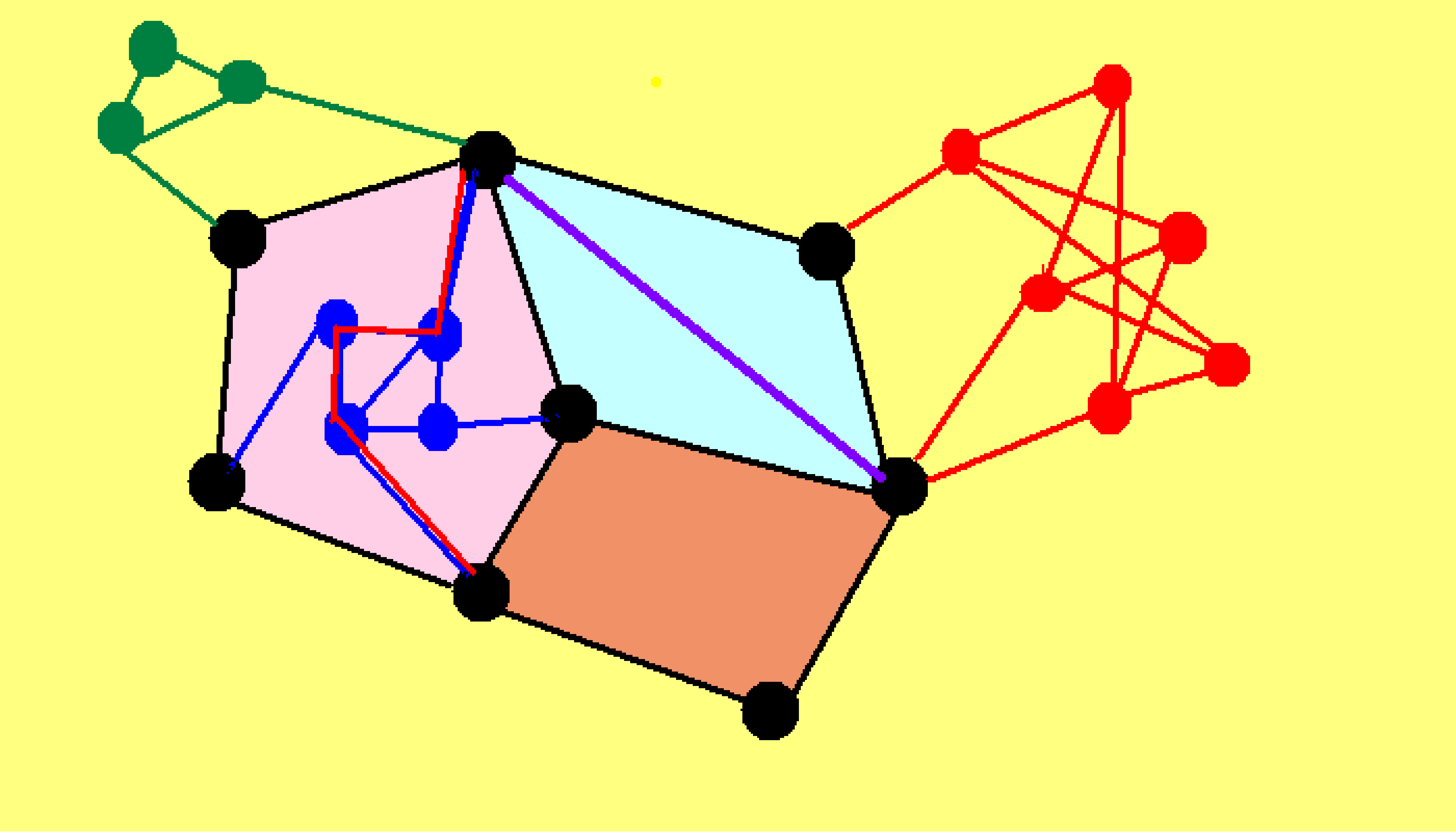


A bridge can be drawn in a face if all its points of attachment lie on that face.

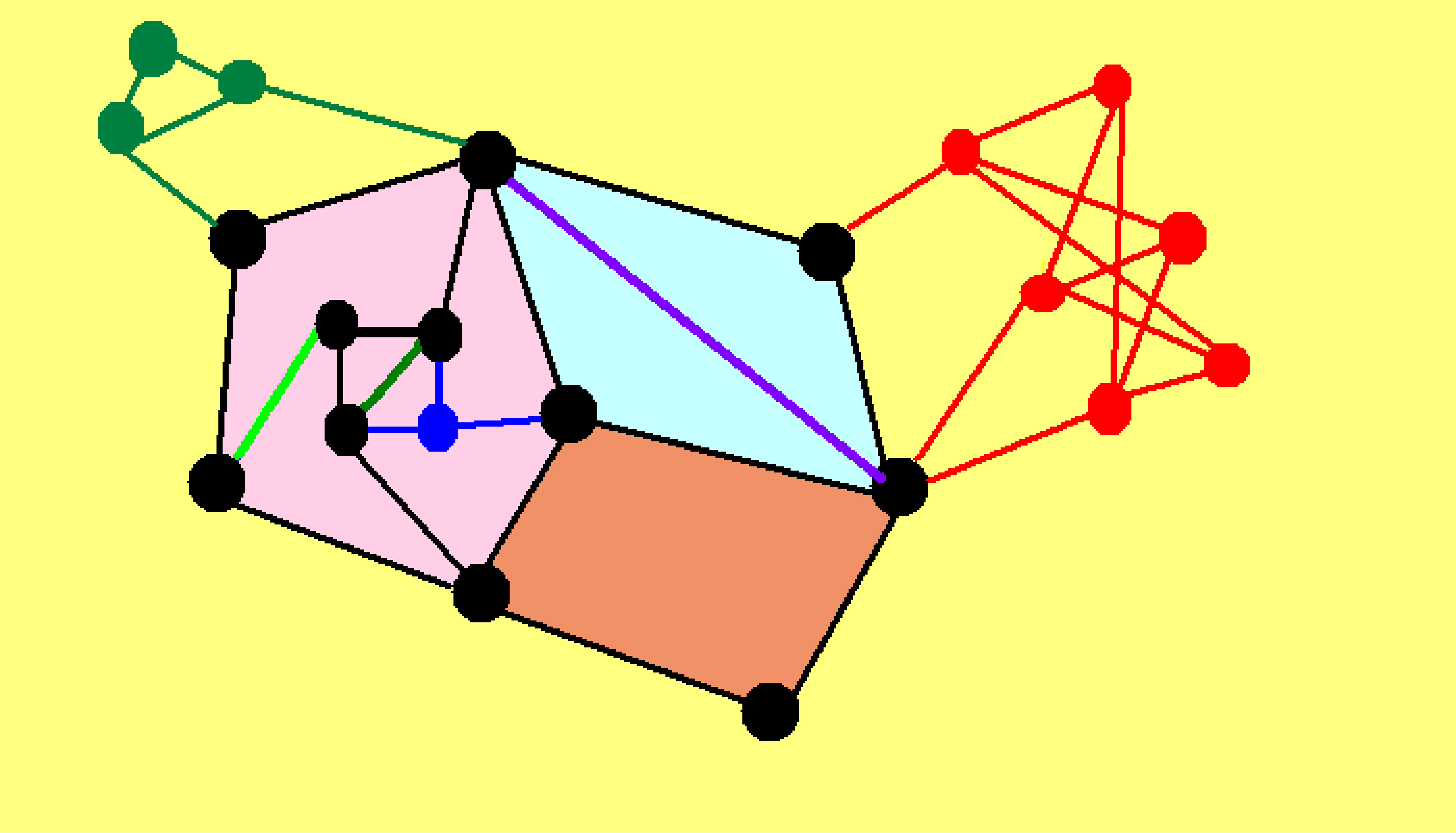


Democron,
Malgrange and
Pertuiset '64:

1. Find a bridge which can be drawn in a minimum number of faces (the blue bridge).



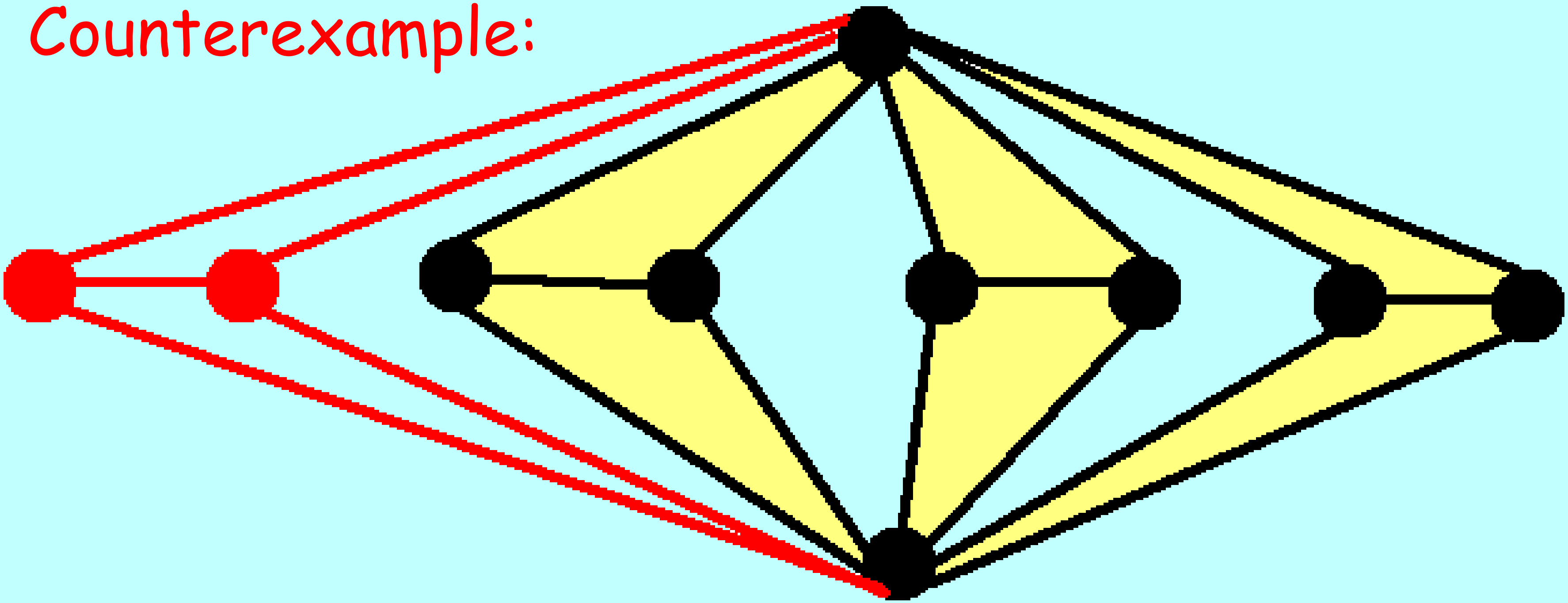
2. Find a path between two points of attachment for that bridge and add the path to the embedding.



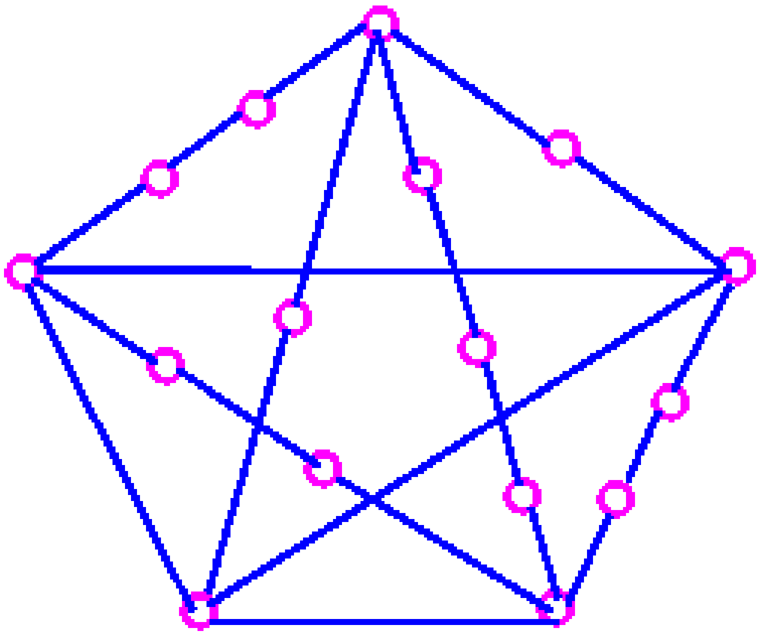
No backtracking required for planarity testing!

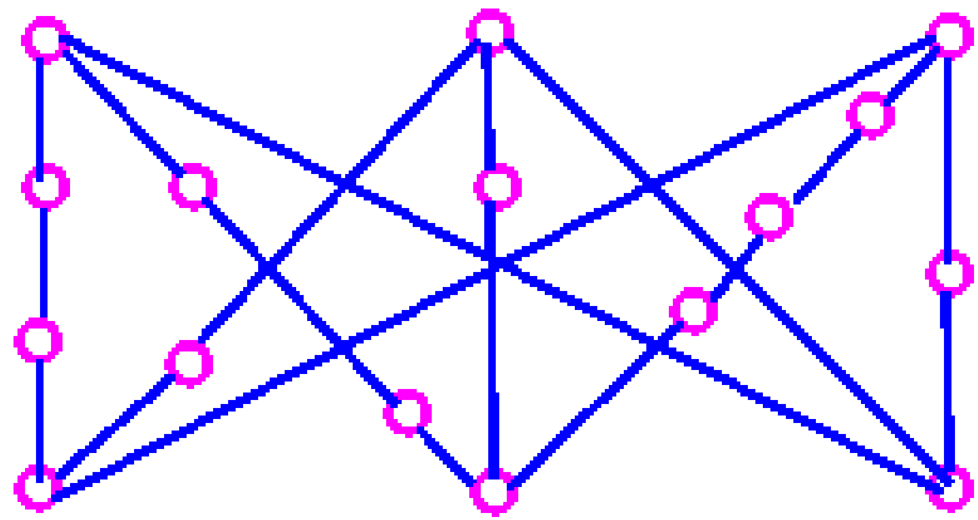
Gibbons: if G is 2-vertex connected, every bridge of G has at least two points of contact and can therefore be drawn in just two faces.

Counterexample:



Graphs homeomorphic to K_5 and $K_{3,3}$:

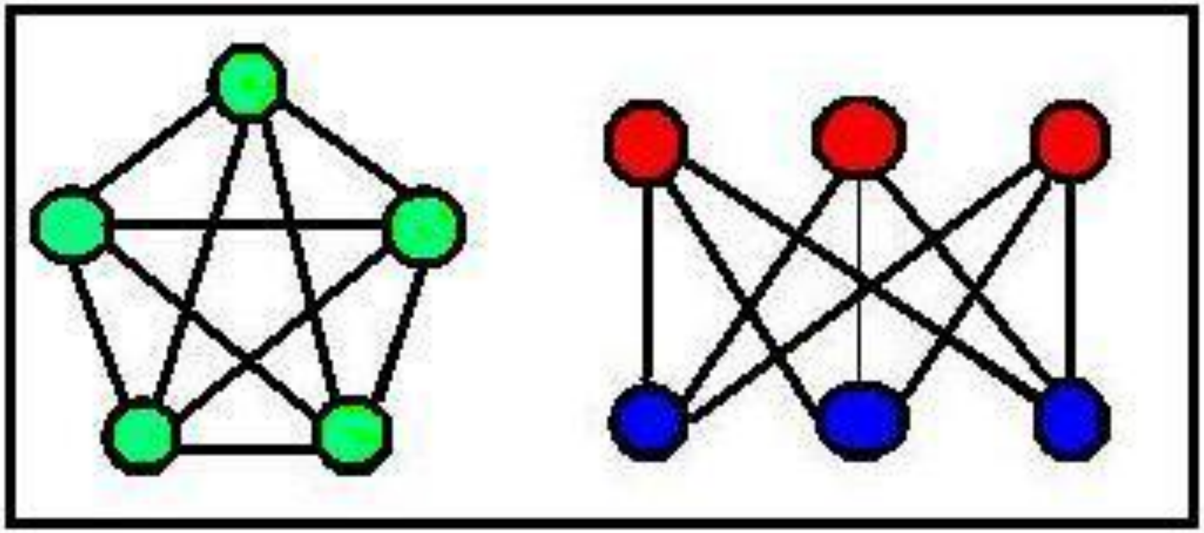




Rashid Bin

Muhammad

Kuratowski's theorem: If G is not planar then it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.



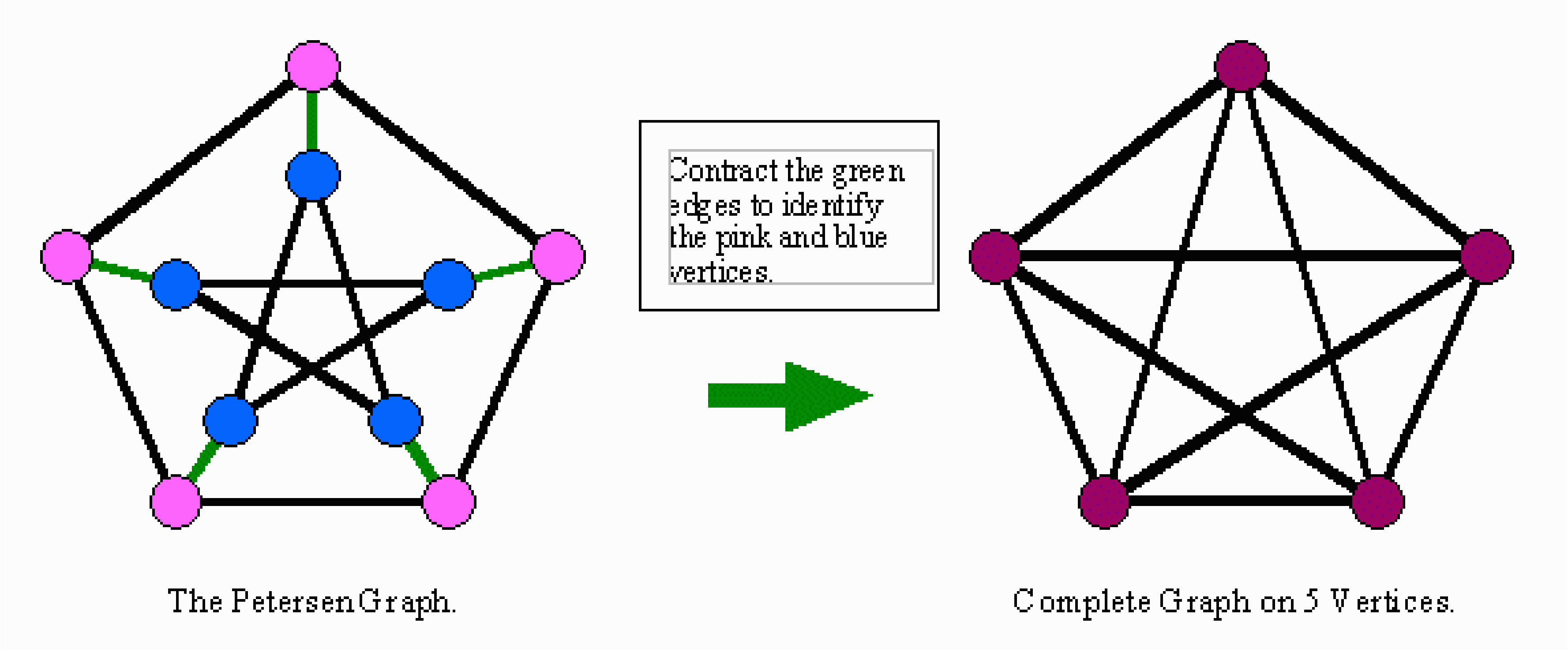
Topological obstruction for surface S:

degrees ≥3, does not embed on S,

G-e embeds on S for all e.

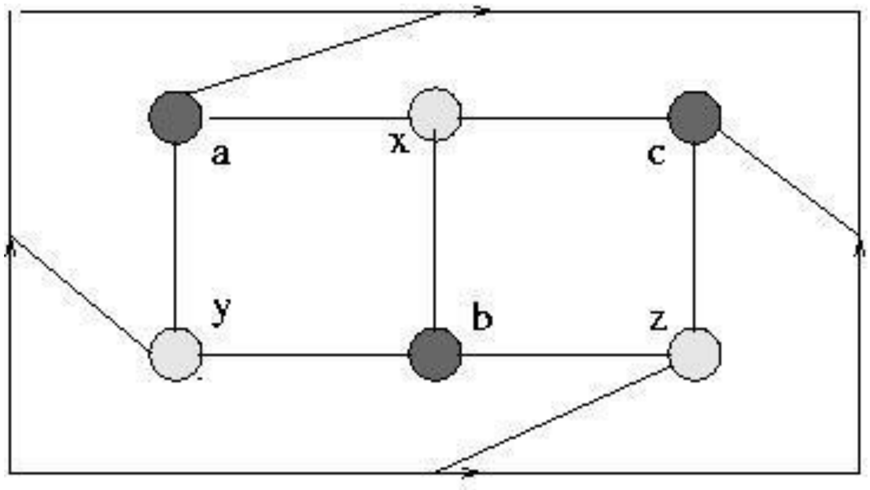
Minor Order Obstruction: Topological obstruction and $G \cdot e$ embeds on S for all e.

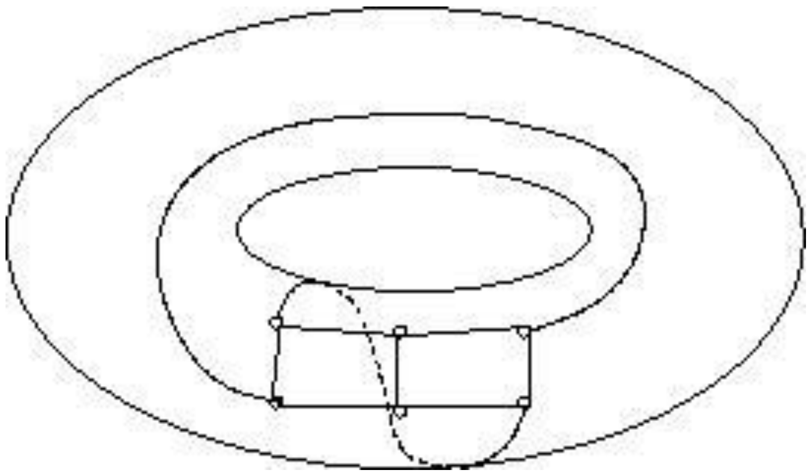
Wagner's theorem: G is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor.



Dale Winter

Torus Embedding



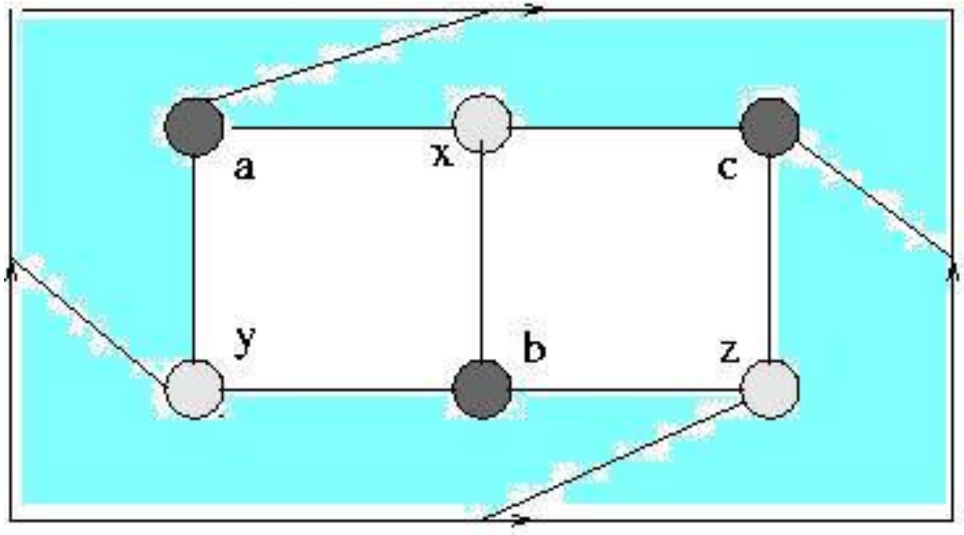


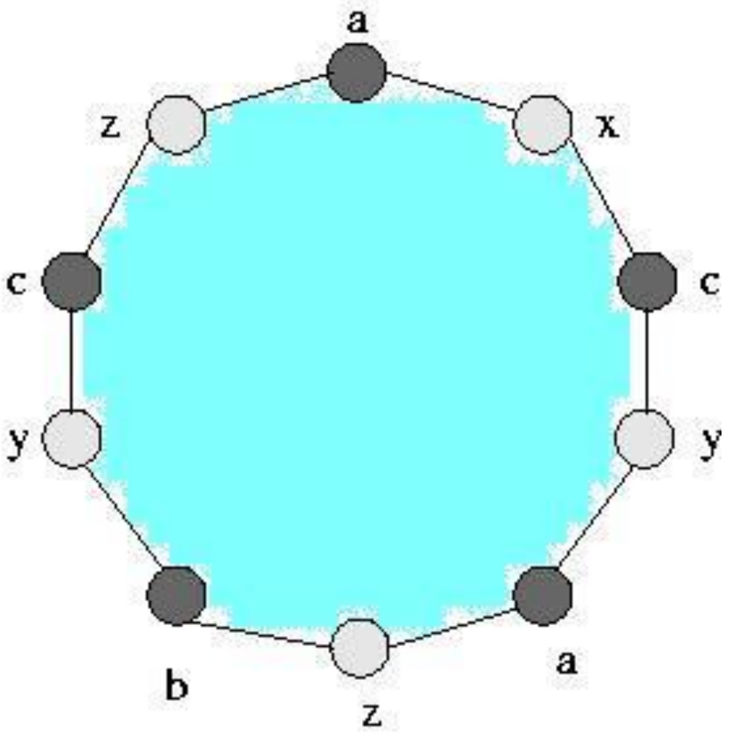
Embedding:

Linear time: Juvan, Marincek & Mohar, '94

$O(n^3)$: Juvan & Mohar, preprint, implementation is buggy

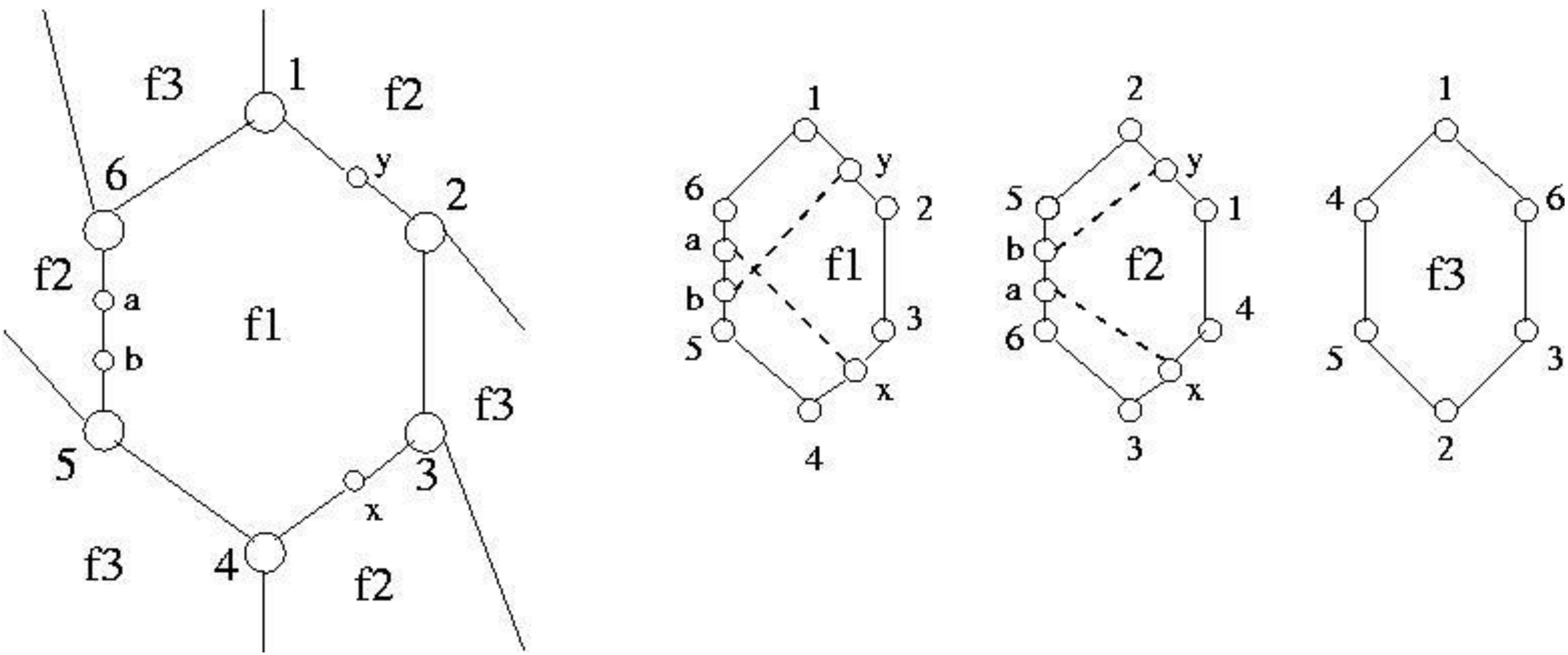
Faces can have repeated vertices and this makes embedding hard:





Indifference Theorem (plane): If B1 and B2 conflict in one face they conflict in all faces.

Does not hold for the torus:



Implemented algorithms run in exponential time:

Myrvold & Neufeld, '96

Woodcock & Myrvold: Jen's thesis '06

For each embedding of K (K_5 or $K_{3,3}$) do

Embed bridges as per Demoucron except:

Choose a minimum penalty bridge at each step

(bridge with path with min # embedding options).

Embed the path in all possible ways.

Gagarin & Kocay '02 + Asano '85: could be used to make the case without a $K_{3,3}$ polynomial time.

Algorithms proved faulty [Kocay & Myrvold]:

I. S. Filotti. An efficient algorithm for determining whether a cubic graph is toroidal. STOC, 1978, pp. 133-142.

I. S. Filotti. An Algorithm for Embedding Cubic Graphs in the Torus. JCSS, volume 2, 1980, pp. 255-276.

I. S. Filotti, G. L. Miller and J. Reif. On determining the genus of a graph in $O(v^{O(g)})$steps. STOC 1979, pp. 27-37.

I.S. Filotti and Jack Mayer. A polynomial algorithm for determining the isomorphism of graphs of fixed genus. STOC 1980, pp. 236-243.

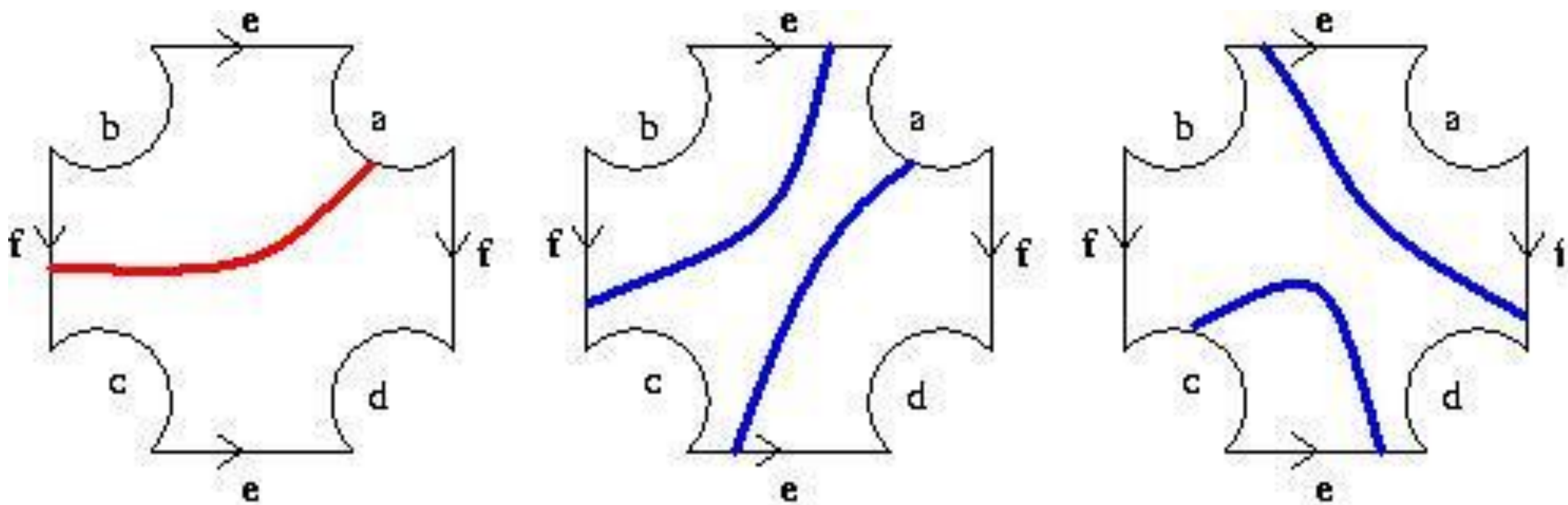
Gary Miller. Isomorphism Testing for Graphs of Bounded Genus. STOC 1980, pp. 225-235.

We shall say that internal chains e and e^{-1} are separated if no two corresponding points on e and e^{-1} are on the same face of g. It is easily seen that e and e^{-1} can be separated in one of three ways:

(i) one chain C from x to y where x is a point of bfc and y is a point of $df^{-1}a$.

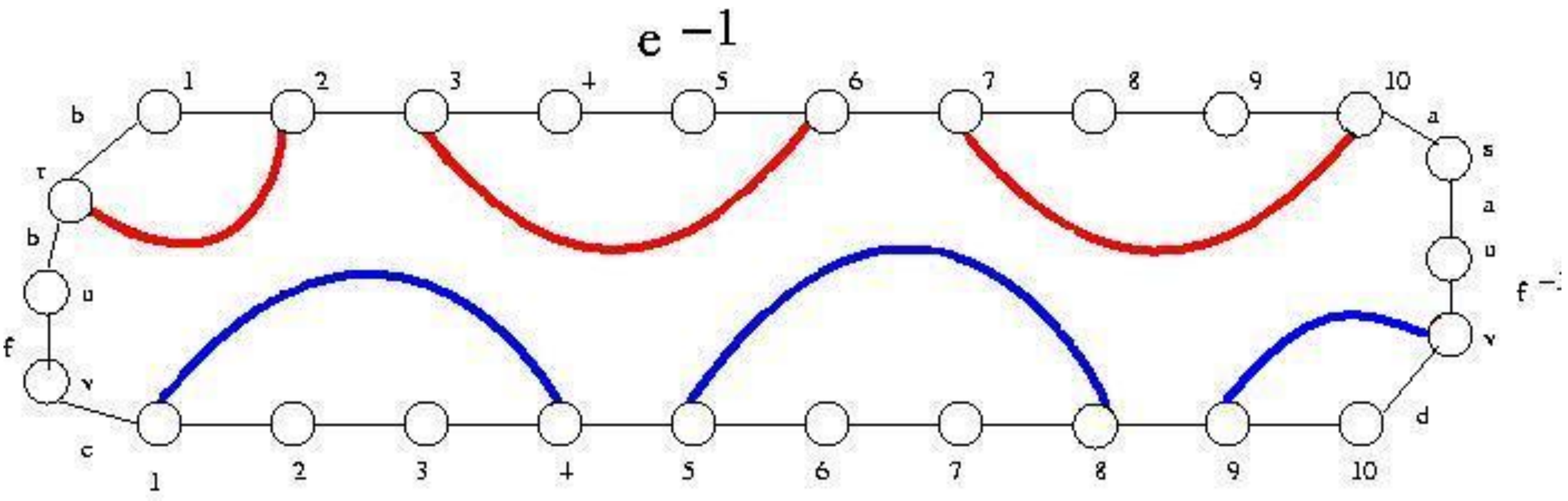
(ii) two chains C1 from x1 to y1 and C2 from x2 to y2 where x1 is a point of bfc, y1 is a point of e, x2 is a point of $df^{-1}a$, and y2 is a point of e^{-1}.

(iii) two chains C1 from x1 to y1 and C2 from x2 to y2 where x1 is a point of $df^{-1}a$, y1 is a point of e, x2 is a point of bfc, and y2 is a point of e^{-1}.



(ii)

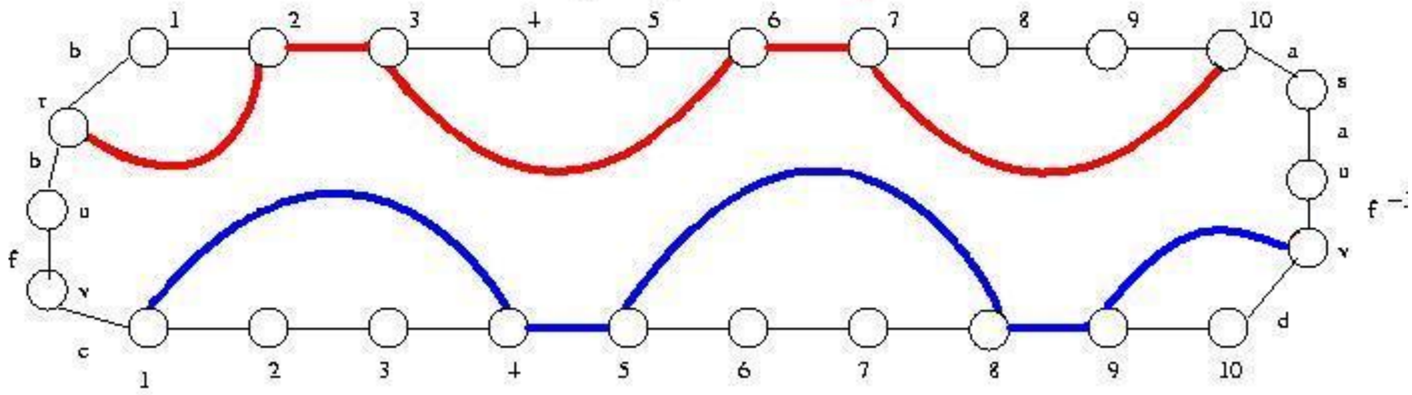
All 6 chains are needed to separate e from e^{-1}:



e= 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

If chains can intersect boundary:

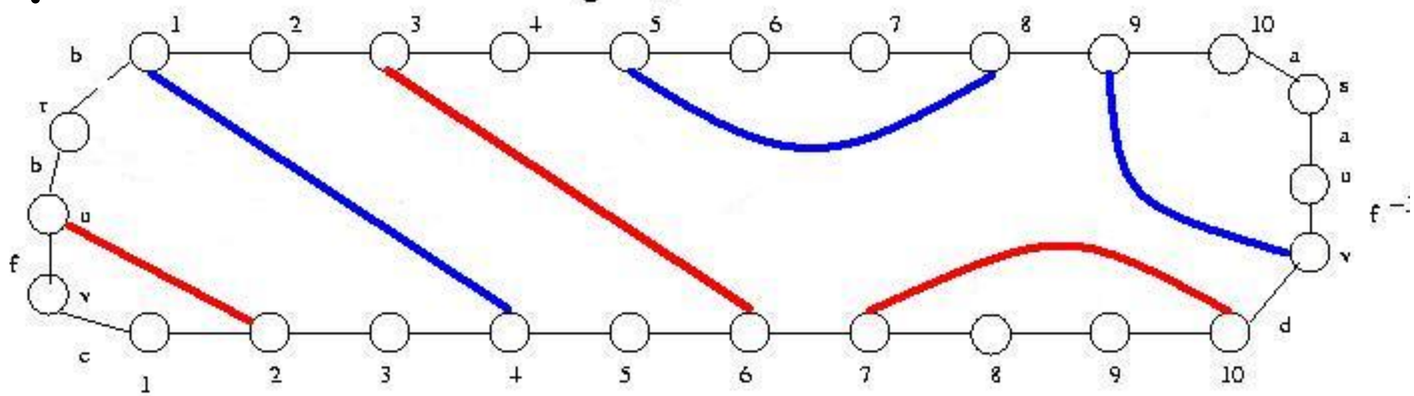
e^{-1}



e= 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

But then we should “embed them in the unique way”.

e^{-1}



e= 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Obstructions for Surfaces

Fact: for any orientable or non-orientable surface, the set of obstructions is finite.

Consequence of Robertson & Seymour theory but also proved independently:

Orientable surfaces: Bodendiek & Wagner, '89

Non-orientable: Archdeacon & Huneke, '89.

How many torus obstructions are there?

8 :	3
9 :	43
10 :	457
11 :	2839
12 :	6426
13 :	5394
14 :	1838
15 :	291
16 :	54
17 :	8
18 :	1

Minor Order Torus Obstructions: 1754

n/m:	18	19	20	21	22	23	24	25	26	27	28	29	30
8 :	0	0	0	0	1	0	1	1	0	0	0	0	0
9 :	0	2	5	2	9	13	6	2	4	0	0	0	0
10 :	0	15	3	18	31	117	90	92	72	17	1	0	1
11 :	5	2	0	46	131	569	998	745	287	44	8	3	1
12 :	1	0	0	52	238	1218	2517	1827	472	79	21	1	0
13 :	0	0	0	5	98	836	1985	1907	455	65	43	0	0
14 :	0	0	0	0	9	68	463	942	222	41	92	1	0
15 :	0	0	0	0	0	0	21	118	43	13	91	5	0
16 :	0	0	0	0	0	0	0	4	3	5	41	0	1
17 :	0	0	0	0	0	0	0	0	0	0	8	0	0
18 :	0	0	0	0	0	0	0	0	0	0	1	0	0

All Torus Obstructions Found So Far:

n/m:	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
8 :	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
9 :	0	2	5	2	9	17	6	2	5	0	0	0	0	0	0	0	0	0	0
10 :	0	15	9	35	40	190	170	102	76	21	1	0	1	0	0	0	0	0	0
11 :	5	2	49	87	270	892	1878	1092	501	124	22	4	1	0	0	0	0	0	0
12 :	1	12	6	201	808	2698	6688	6372	1933	482	94	6	2	0	0	0	0	0	0
13 :	0	0	12	19	820	4967	12781	16704	7182	1476	266	52	1	0	0	0	0	0	0
14 :	0	0	0	9	38	2476	15219	24352	16298	3858	808	215	19	0	0	0	0	0	0
15 :	0	0	0	0	0	33	3646	22402	20954	8378	1859	708	184	5	0	0	0	0	0
16 :	0	0	0	0	0	0	20	2689	17469	10578	3077	1282	694	66	1	0	0	0	0
17 :	0	0	0	0	0	0	0	0	837	8099	4152	1090	1059	368	11	0	0	0	0
18 :	0	0	0	0	0	0	0	0	0	133	2332	1471	511	639	102	1	0	0	0
19 :	0	0	0	0	0	0	0	0	0	0	0	393	435	292	255	15	0	0	0
20 :	0	0	0	0	0	0	0	0	0	0	0	0	39	100	164	63	2	0	0
21 :	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	63	1	0	0
22 :	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	22	0	0
23 :	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0
24 :	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2