$$
\begin{aligned}
X5 &= b_5 - a_{5,1}\,X1 - a_{5,2}\,X2 - a_{5,3}\,X3 - a_{5,4}\,X4 \\
X6 &= b_6 - a_{6,1}\,X1 - a_{6,2}\,X2 - a_{6,3}\,X3 - a_{6,4}\,X4 \quad D^* \\
X7 &= b_7 - a_{7,1}\,X1 - a_{7,2}\,X2 - a_{7,3}\,X3 - a_{7,4}\,X4
\end{aligned}
$$

---------------------------------------------

$$Z = v + c_1\,X1 + c_2\,X2 + c_3\,X3 + c_4\,X4$$

If X1=X2=X3=X4=0, X5=3, X6=5, X7=7, z=4 is a solution to D*, what does this say about D*?

If in addition, D* has a solution:
X1=X2=X4=0, X3=1, X5=1, X6=7, X7=3, z=6

what else can you conclude about D*?

Last class: degenerate pivoting caused an infinite loop.

Theorem [Bland, 1977] The Simplex method does not repeat dictionaries (and hence terminates) as long as both the entering and leaving variables are chosen by the smallest-subscript rule in each iteration.

Note about reading a mathematical proof:
It’s not like reading a novel.

You sometimes have to read a proof very slowly writing down and checking every step as you go.

It took me a long time to believe and follow the next proof.

Fortunately, most of the proofs in this class are not this tedious.

We need to show cycling does not happen. Suppose it does:

$D_0, D_1, D_2, D_3, \ldots, D_k = D_0, D_1, D_2, D_3, \ldots, D_k = D_0,$

…

Fickle variable: basic in some of these dictionaries and non-basic in others.

$x_t$ = fickle variable with largest subscript.

Idea of proof: *Argue* that when $x_t$ leaves, some variable $x_r$ with $r < t$ was eligible to leave and should have left instead.

Pivot from dictionary D: $x_t$ leaves and $x_s$ enters ($x_t$ is basic in D but not in the next dictionary).

Pivot from dictionary D*: $x_t$ enters again.

Technical point:
Since the loop is cyclic:
$D_0, D_1, D_2, D_3, \ldots, D_k{=}D_0, D_1, D_2, D_3, \ldots, D_k{=}D_0,$

D$=$ $D_i$, and D* $=$ $D_j$ but we may have $j < i$.

Dictionary D ($x_t$ leaves and $x_s$ enters) :

$x_i = b_i \quad - \quad \sum_{j \notin B} a_{ij}\, xj \qquad$ for each $i \in B$.

---------------------------------------------------------------

$z = v \quad + \quad \sum_{j \notin B} c_j\, xj$

Since all the pivots are degenerate, the last row of D* has the same constant term v. The last row of D* (including all variables):

$z = v \quad + \quad \sum_{j=1}^{n+m} c_j^{\;*}\; xj$

Set $c_j^* = 0$ if $x_j$ is basic in D*.

Dictionary D:

$x_i = b_i \quad - \quad \sum_{j \notin B} a_{ij}\, xj \qquad$ for each $i \in B$.

-----------------------------------------------------------

$z = v \quad + \quad \sum_{j \notin B} c_j\, xj$

One solution to D:

Set non-basic variable (the entering one) $x_s = y$ and the other non-basic ones to 0 and get the rest from the dictionary:

$x_i = b_i \quad - a_{is}\ y \quad$ for each $i \in B$.

------------------------------------------------------------

$z = v \quad + c_s\ y$

Set non-basic variable (the entering one) $x_s = y$ and the other non-basic ones to 0 and get the rest from the dictionary:

$x_i = b_i \quad - a_{is} \; y$ for each $i \in B$.

---------------------------------------------------------

$z = v \quad + c_s \; y$

Plug this into equation for z from D*:

$z = v \quad + \sum_{j=1}^{n+m} c_j^{\;*} \; xj$ and hence:

$z = v \quad + c_s^{*}\, y + \sum_{i \in B} c_i^{\;*} \left( b_i \quad - a_{is} \; y \right)$

The other variables have value 0 so I have not included them in the z equation here.

From D: $z = v + c_s\, y$

From D* (same value for z):

$z = v + c_s^* y + \sum_{i \in B} c_i^* (b_i - a_{is}\, y)$

So:

$v + c_s\, y = v + c_s^* y + \sum_{i \in B} c_i^* (b_i - a_{is}\, y)$

Simplify algebraically:

$(c_s - c_s^* + \sum_{i \in B} c_i^* a_{is})\, y = \sum_{i \in B} c_i^* b_i$

$$(c_s - c_s^* + \sum_{i \in B} c_i^* a_{is})\, y = \sum_{i \in B} c_i^* b_i$$

The RHS is a constant that does not depend on y! So how can this be satisfied for all y?

Only when:

$$(c_s - c_s^* + \sum_{i \in B} c_i^* a_{is}) = 0$$

$(c_s - c_s^* + \sum_{i \in B} c_i^* a_{is}) = 0$

Since $x_s$ enters when we pivot from D:
$c_s > 0$. Since $x_s$ does not enter when we pivot from D* ($x_t$ enters and $t > s$, so $x_s$ is not eligible to enter), $c_s^* \le 0$.
Therefore $c_s - c_s^* > 0$. Hence for at least one value of r with $r \in B$, $c_r^* a_{rs} < 0$.
Since $r \in B$, $x_r$ is basic in D.
Since $c_r^*$ is not 0, $x_r$ is not basic in D*.
Hence, $x_r$ is fickle and $r \le t$.

Can we have $x_r = x_t$? No!

$x_t$ leaves from D and $x_s$ enters.

This means that D has a row like this:

$x_t = \ 0 \ - \ a_{ts} x_s + ....$

where we must have $a_{ts} > 0$.

Since t enters from D*, $c_t^* > 0$.

Therefore, $c_t^* a_{ts} > 0$.

But recall that we have: $c_r^* a_{rs} < 0$.

Therefore r and t are different.

Now r< t and r does not enter in D*.
Therefore $c_r^* \leq 0$.
But recall that we have: $c_r^* a_{rs} < 0$.
So $a_{rs} > 0$.
Fickle variables stay 0 over all the degenerate pivots: if one of them increased then z would be able to increase. So $x_r=0$ in D and D*.
D must have had a row like this:
$x_r = 0 - a_{rs} x_s + \ldots$ with $a_{rs} > 0$
so r should have exited from D instead of t when s entered.
This is the contradiction we need to end the proof.

## Observation (from the proof):

If you use smallest subscript rule to decide on both the entering and exiting variables:

Consider all the variables that are fickle in the degenerate pivots: once the one with maximum subscript leaves then it is not permitted to come back again until you have some non-degenerate pivot.