How fast is the Simplex method? Is it polynomial time when it is implemented to not cycle?

Klee-Minty, n=2

```
Maximize
10 X1 + 1 X2
subject to
     1 X1 + 0 X2  <=   1
    20 X1 + 1 X2  <= 100

X1 , X2 >= 0
```

The initial dictionary:

```
X3 =   1  - 1 X1 + 0 X2
X4 =  100 -20 X1 - 1 X2
-------------------------
z  =  -0  +10 X1 + 1 X2
```

X1 enters. X3 leaves. z = 0

After 1 pivot:

```
X1 =   1 + 0 X2 - 1 X3
X4 =  80 - 1 X2 +20 X3
-------------------------
z  =  10 + 1 X2 -10 X3
```

X2 enters. X4 leaves. z = 10

After 2 pivots:

```
X1 =   1 - 1 X3 + 0 X4
X2 =  80 +20 X3 - 1 X4
-----------------------
z  =  90 +10 X3 - 1 X4
```

X3 enters. X1 leaves. z = 90

After 3 pivots:

```
X3 =    1 - 1 x1 + 0 X4
X2 =  100 -20 X1 - 1 X4
------------------------
z  =  100 -10 X1 - 1 X4
```

The optimal solution: 100

Klee-Minty, n=3

Maximize

$100 X1 + 10 X2 + 1 X3$

subject to

$$
\begin{aligned}
1 X1 + 0 X2 + 0 X3 &\le 1 \\
20 X1 + 1 X2 + 0 X3 &\le 100 \\
200 X1 + 20 X2 + 1 X3 &\le 10000
\end{aligned}
$$

$X1 , X2 , X3 \ge 0$

Using the maximum coefficient rule:

The initial dictionary:

```
X4 =       1 -  1 X1 + 0 X2 + 0 X3
X5 =     100 - 20 X1 - 1 X2 + 0 X3
X6 =   10000 -200 X1 -20 X2 - 1 X3
----------------------------------
z  =   0 +    100 X1 +10 X2 + 1 X3

X1  enters.   X4  leaves.  z = -0
```

After   1 pivot:

```
X1 =      1 +  0 X2 + 0 X3 -  1 X4
X5 =     80 -  1 X2 + 0 X3 + 20 X4
X6 =   9800 - 20 X2 - 1 X3 +200 X4
----------------------------------
z  =    100  +10 X2 + 1 X3 -100 X4
```

```
After   2 pivots:
X1 =      1 + 0 X3 -  1 X4 + 0 X5
X2 =     80 + 0 X3 + 20 X4 - 1 X5
X6 =   8200 - 1 X3 -200 X4 +20 X5
----------------------------------
z  =   900  +  1 X3 +100 X4 -10 X5

X4  enters.   X1  leaves.  z = 900
After   3 pivots:
X4 =      1 -  1 X1 + 0 X3 + 0 X5
X2 =    100 - 20 X1 + 0 X3 - 1 X5
X6 =   8000 +200 X1 - 1 X3 +20 X5
-----------------------------------
z  =   1000 -100 X1 + 1 X3 -10 X5
X3  enters.   X6  leaves.  z = 1000
```

After 4 pivots:

```
X4 =     1 -  1 X1 + 0 X5 + 0 X6
X2 =   100 - 20 X1 - 1 X5 + 0 X6
X3 =  8000 +200 X1 +20 X5 - 1 X6
-----------------------------------
z  =  9000 +100 X1 +10 X5 - 1 X6
```

X1 enters. X4 leaves. z = 9000

After 5 pivots:

```
X1 =     1 -  1 X4 + 0 X5 + 0 X6
X2 =    80 + 20 X4 - 1 X5 + 0 X6
X3 =  8200 -200 X4 +20 X5 - 1 X6
---------------------------------
z  =  9100 -100 X4 +10 X5 - 1 X6
```

X5 enters. X2 leaves. z = 9100

After 6 pivots:

```
X1 =      1 + 0 X2 -   1 X4 + 0 X6
X5 =     80 - 1 X2 + 20 X4 + 0 X6
X3 =   9800 -20 X2 +200 X4 - 1 X6
---------------------------------
z  =   9900 -10 X2 +100 X4 - 1 X6
```

X4 enters. X1 leaves. z = 9900

After 7 pivots:

```
X4 =      1 -   1 X1 + 0 X2 + 0 X6
X5 =    100 -  20 X1 - 1 X2 + 0 X6
X3 =  10000 -200 X1 -20 X2 - 1 X6
-----------------------------------
z  =  10000 -100 X1 -10 X2 - 1 X6
```

The optimal solution: 10000

Klee-Minty, n=4

Maximize 1000 X1 +100 X2 +10 X3 + 1 X4

subject to

```
   1 X1 +  0 X2 + 0 X3 + 0 X4 ≤ 1
  20 X1 +  1 X2 + 0 X3 + 0 X4 ≤ 100
 200 X1 + 20 X2 + 1 X3 + 0 X4 ≤ 10000
2000 X1 +200 X2 +20 X3 + 1 X4 ≤ 1000000
```

X1 , X2 , X3 , X4 >= 0

After 15 pivots:

```
X5=       1    - 1 X1+  0 X2+ 0 X3+ 0 X8
X6=     100    -20 X1-  1 X2+ 0 X3+ 0 X8
X7=   10000   -200 X1- 20 X2- 1 X3+ 0 X8
X4= 1000000 -2000 X1-200 X2-20 X3- 1 X8
---------------------------------------
z = 1000000 -1000 X1-100 X2-10 X3- 1 X8
```

```
The optimal solution: 1000000
X1  =  0,  X2  =  0,  X3  =  0,
X4  = 1000000,
X5  =  1,  X6  = 100,
X7  = 10000,  X8  =  0
```

The General Problem:

Maximize

$$10^{n-1} x_1 + 10^{n-2} x_2 + 10^{n-3} x_3 + \ldots + 10^{n-n} x_n$$

subject to

$$
\begin{aligned}
X_1 &\le 1 \\
20 X_1 + X_2 &\le 100 \\
200 X_1 + 20 X_2 + X_3 &\le 10000 \\
2000 X_1 + 200 X_2 + 20 X_3 + X_4 &\le 1000000
\end{aligned}
$$

...

$$X_1, X_2, X_3, \ldots, X_n \ge 0$$

Theorem [Klee-Minty, 1972] The Klee-Minty examples take $2^n - 1$ iterations when the variable to enter is chosen using the *Maximum Coefficient* rule.

Proof: Problems 4.2 and 4.3.

Similar examples exist for *Largest Increase rule* [Jeroslow, 1973].

So why is the Simplex Method useful? In practice, it usually takes less than 3m/2 iterations, and only rarely 3m, for $m < 50$, and $m+n < 200$ [Dantzig, 1963].

Monte Carlo studies of larger random problems- similar results, see table in text [Kuhn and Quandt, 1963].

The Largest Increase rule may require fewer iterations but it requires more work per iteration. Thus, the Maximum Coefficient rule may be faster.

Application of Maximum Increase Rule: Dictionary:

$x_i \quad = \quad b_i \quad - \quad s * x_j \quad + \ ....$

If x_j enters and x_i leaves:

$s\ x_j = \quad b_i \qquad - \ x_i \ + \ ...$

So $x_j \ = \ b_i/s \quad - \ x_i \ + \ ...$

Looking at the z row:

$z = z' \quad + \quad r\ Xj$

So $x_j = b_i/s - x_i + \ldots$

Looking at the z row:

$z = z' + r\,x_j$

Replacing x_j in the z row:

$z = z' + r\,(b_i / s) - \ldots$

Change to z is $(r/s) * b_i$ where r is the coefficient of the entering variable in the z row, b_i is the constant term in the pivot row, and $-s$ is the coefficient of x_j in the pivot row of the dictionary

Largest Increase Rule: Choose the entering variable to maximize this.