What happens to Klee-Minty examples if maximum increase rule is used?

Klee-Minty, n=2

Maximize
10 X1 + 1 X2
subject to

$$1\,X1 + 0\,X2 \le 1$$
$$20\,X1 + 1\,X2 \le 100$$

$$X1,\ X2 \ge 0$$

Last class:

Original phase 1 : Minimize $z_1 = x_0$

Standard form phase 1: Maximize $z_2 = -x_0$

Algebraically equivalent formula:

z_2 = -7+ 1 X1 - 1 X2 + 0 X3 + 0 X4 - 1 X6

Objective function used for new problem:

z_3 = 1 X1 - 1 X2 + 0 X3 + 0 X4 - 1 X6

Renumbering:

z_3 = 1 X1 - 1 X2 + 0 X3 + 0 X4 - 1 X5

Note that $z_3 = z_2 + 7$.

We want $x_0 = 0$ which means $z_1 = 0 \implies z_2 = 0$ for a feasible problem. When $z_2 = 0$, $z_3 = 7$.

Before pivoting to make feasible:

```
X5 = -3 + X1          -   X3- 2 X4 +  X0
X6 = -7 + X1  -  X2             +  X0
X7 = -5 + X1  -2 X2        -   X4 +  X0
-------------------------------------
z =  0                          -  X0
```

Taking the first pivot:

```
X0 enters.  X6 leaves. z = -0.000000

The initial dictionary:
X5 = 4+ 0 X1 + 1 X2 - 1 X3 - 2 X4 + 1 X6
X0 = 7- 1 X1 + 1 X2 + 0 X3 + 0 X4 + 1 X6
X7 = 2+ 0 X1 - 1 X2 + 0 X3 - 1 X4 + 1 X6
-------------------------------------
z = -7+ 1 X1 - 1 X2 + 0 X3 + 0 X4 - 1 X6
```

The initial dictionary for z_3:

```
X6 = 4+0 X1 +1 X2 -1 X3 -2 X4 +1 X5
X7 = 7-1 X1 +1 X2 +0 X3 +0 X4 +1 X5
X8 = 2+0 X1 -1 X2 +0 X3 -1 X4 +1 X5
-------------------------------------
```

z_3 = -0+1 X1 -1 X2 +0 X3 +0 X4 -1 X5

X1 enters. X7 leaves. z = -00000

After 1 pivot:

```
X6 = 4+1 X2 -1 X3 -2 X4 +1 X5 +0 X7
X1 = 7+1 X2 +0 X3 +0 X4 +1 X5 -1 X7
X8 = 2-1 X2 +0 X3 -1 X4 +1 X5 +0 X7
-------------------------------------
```

z_3 = 7+0 X2 +0 X3 +0 X4 +0 X5 -1 X7

Renumber this:

X6 = 4+1 X2 -1 X3 -2 X4 +1 X5 +0 X7

X1 = 7+1 X2 +0 X3 +0 X4 +1 X5 -1 X7

X8 = 2-1 X2 +0 X3 -1 X4 +1 X5 +0 X7

----

z_3 = 7+0 X2 +0 X3 +0 X4 +0 X5 -1 X7

In terms of the original problem:

X5 = 4+1 X2 -1 X3 -2 X4 +1 X6 +0 X0

X1 = 7+1 X2 +0 X3 +0 X4 +1 X6 -1 X0

X7 = 2-1 X2 +0 X3 -1 X4 +1 X6 +0 X0

Original objective function:

Maximize $X_1 + X_2 + 2\ X_3 + X_4$

Assignment 2:
Due at beginning of class on Fri. Oct. 3.
Late submissions accepted with 10% penalty until Tues. Oct. 7 at 12:30pm.

Project description for CSC 545:
Due on Fri. Oct. 17 at 11:55pm.

are both available now from our web pages but not yet on connex (coming soon).

Our midterm is in class on Fri. Oct. 10.

What happens to Klee-Minty examples if maximum increase rule is used?

Klee-Minty, n=2

Maximize

$10 X1 + 1 X2$

subject to

$$1 X1 + 0 X2 \le 1$$
$$20 X1 + 1 X2 \le 100$$

$X1 , X2 \ge 0$

The initial dictionary:

```
X3 =   1  - 1 X1 + 0 X2
X4 =  100 -20 X1 - 1 X2
-------------------------
z  =   0  +10 X1 + 1 X2
```

If X1 enters, how much will z increase?

If X2 enters, how much will z increase?

```
X1 enters and X3 leaves:
the increase to z is:  10.00
X2 enters and X4 leaves:
the increase to z is: 100.00

X2  enters. X4 leaves.

After   1 pivot: (Before 3 pivots)

X3 =  1.00-  1.00 X1 +  0.00 X4
X2 =100.00- 20.00 X1 -  1.00 X4
---------------------------------
z  =100.00- 10.00 X1 -  1.00 X4

The optimal solution: 100.000000
```

Klee-Minty, n=3

Maximize

$$100\,X1 + 10\,X2 + 1\,X3$$

subject to

$$
\begin{aligned}
1\,X1 + 0\,X2 + 0\,X3 &\le 1 \\
20\,X1 + 1\,X2 + 0\,X3 &\le 100 \\
200\,X1 + 20\,X2 + 1\,X3 &\le 10000
\end{aligned}
$$

$$X1\,,\ X2\,,\ X3 \ge 0$$

The initial dictionary:

```
X4 =      1 -   1 X1 + 0 X2 + 0 X3
X5 =    100 -  20 X1 - 1 X2 + 0 X3
X6 =  10000 -200 X1 -20 X2 - 1 X3
----------------------------------
z  =  0 +    100 X1 +10 X2 + 1 X3
```

If X1 enters, how much will z increase?

If X2 enters, how much will z increase?

If X3 enters, how much will z increase?

The initial dictionary:

```
X4 =       1 -   1 X1 + 0 X2 + 0 X3
X5 =     100 -  20 X1 - 1 X2 + 0 X3
X6 =   10000 -200 X1 -20 X2 - 1 X3
---------------------------------
z  =     0 +    100 X1 +10 X2 + 1 X3

X1: the increase to z is: 100
X2: the increase to z is: 1000
X3: the increase to z is: 10000

X3  enters.   X6  leaves.  z = 0.00
```

```
After   1 pivot: (Before 7 pivots)
X4 =    1-  1 X1 + 0 X2 + 0 X6
X5 =  100- 20 X1 - 1 X2 + 0 X6
X3 =10000-200 X1 -20 X2 - 1 X6
--------------------------------
z  =10000-100 X1 -10 X2 - 1 X6

The optimal solution: 10000
```

Theorem [Klee-Minty, 1972] The Klee-Minty examples take 2^n - 1 iterations when the variable to enter is chosen using the *Maximum Coefficient* rule.

Proof: Problems 4.2 and 4.3.

Similar examples exist for *Largest Increase rule* [Jeroslow, 1973].

So why is the Simplex Method useful? In practice, it usually takes less than 3m/2 iterations, and only rarely 3m, for $m < 50$, and $m+n < 200$ [Dantzig, 1963].

Monte Carlo studies of larger random problems- similar results, see table in text [Kuhn and Quandt, 1963].

The Largest Increase rule may require fewer iterations but it requires more work per iteration. Thus, the Maximum Coefficient rule may be faster.

Application of Maximum Increase Rule: Dictionary:

$x_i \;=\; b_i \;-\; s * x_j \;+ \;....$

If x_j enters and x_i leaves:

$s\ x_j = \; b_i \; - x_i + \; ...$

So $x_j \;=\; b_i/s \;-\; x_i/s + \;...$

Looking at the z row:

$z = v \;+\; r\ x_j$

So $x_j = b_i/s - x_i/s + ...$

Looking at the z row:

$z = v + r\, x_j + \ldots$

Replacing x_j in the z row:

$z = v + r\, (b_i / s) + ...$

Change to z is $(r/s) * b_i$ where r is the coefficient of the entering variable in the z row, b_i is the constant term in the pivot row, and -s is the coefficient of x_j in the pivot row of the dictionary

Largest Increase Rule: Choose the entering variable to maximize this.

Maximize

$$5 X1 + 2 X2 + 4 X3 + 3 X4$$

subject to

$$-1 X1 + 1 X2 + 1 X3 + 2 X4 \leq 2$$
$$1 X1 + 2 X2 + 0 X3 + 3 X4 \leq 8$$

$$X1 , X2 , X3, X4 \geq 0$$

If I told you the answer had X1 and X3 in the basis, how can you more directly find the final dictionary?

```
-1 X1 + 1 X2 + 1 X3 + 2 X4  ≤  2
 1 X1 + 2 X2 + 0 X3 + 3 X4  ≤  8
```

In matrix format with the slacks:

$$\begin{bmatrix} -1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 3 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

When we are done we need to have x_1 and x_3 in the basis:

$$\begin{bmatrix} 1 & * & 0 & * & * & * \\ 0 & * & 1 & * & * & * \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 3 & 0 & 1 \end{bmatrix} x = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$\begin{bmatrix} 1 & * & 0 & * & * & * \\ 0 & * & 1 & * & * & * \end{bmatrix} x = \begin{pmatrix} * \\ * \end{pmatrix}$$

To get here, multiply by B^{-1}

where $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

To get here, multiply by B^{-1}

where $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

B^{-1}

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 3 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 1 \\ 0 & 3 & 1 & 5 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 1 \\ 0 & 3 & 1 & 5 & 1 & 1 \end{bmatrix} x = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$$

Recall that $B^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Last dictionary(reordered by subscript):

```
X1 =  8- 2 X2 - 3 X4 + 0 X5 - 1 X6
X3 = 10- 3 X2 - 5 X4 - 1 X5 - 1 X6
---------------------------------------
z  = 80-20 X2 -32 X4 - 4 X5 - 9 X6
```

-1 * B^{-1} is hiding in the dictionary (slack coefficients).

Recall that one way given a matrix B to find B^{-1} is to transform: $[\,B \mid I\,]$ into $[\,I \mid B^{-1}\,]$.

The slack portion of the matrix starts out as I and so it ends up as B^{-1}.

Because we rearrange the equations so that the non-basic variables are on the other side of the equation, we see -1 times the inverse in the dictionary.