Consider this linear programming problem:

Maximize $z = x_1 + x_2$

subject to

$$\begin{aligned} x_1 \quad\quad\quad &\leq 3 \\ x_2 &\leq 3 \\ 2x_1 + 2x_2 &\leq 8 \end{aligned}$$

$x_1, x_2 \geq 0$

1. Draw a picture that shows these constraints.
2. Indicate the feasible region by shading it in.
3. At each corner of the feasible region, indicate the values of $x_1$, $x_2$, and z.
4. Characterize the optimal solutions to the problem by examining the picture.

Consider this linear programming problem:

Maximize $z = x_1 + x_2$

subject to

$$\begin{aligned} x_1 \quad\quad\quad &\le 3 \\ x_2 &\le 3 \\ 2x_1 + 2x_2 &\le 8 \end{aligned}$$

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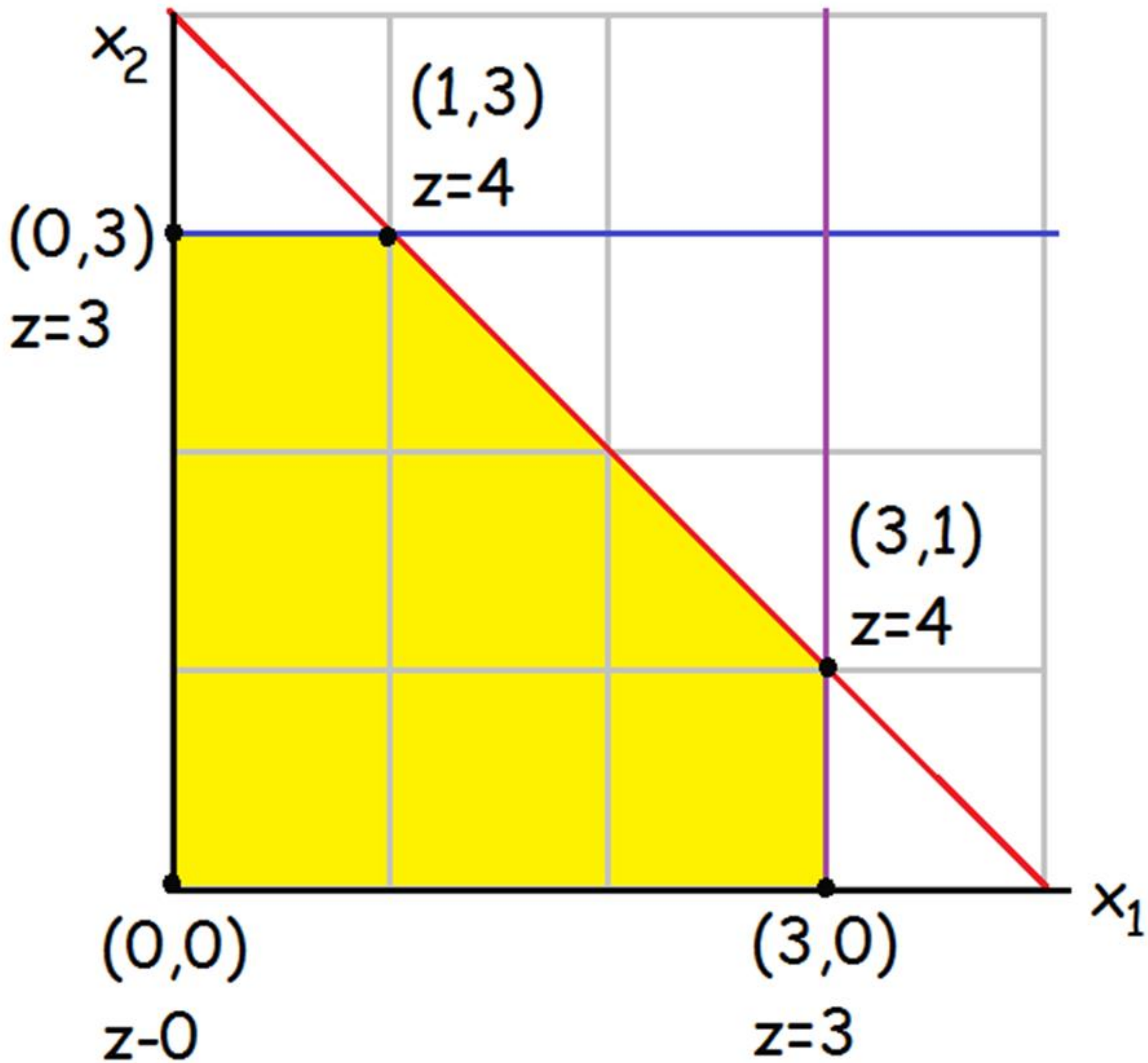

Assignment 2: Due at beginning of class on Fri. Oct. 3. Late submissions accepted with 10% penalty until Tues. Oct. 7 at 12:30pm.

Project description for CSC 545:
Due on Fri. Oct. 17 at 11:55pm.

Team lists are posted on the class web page.

Our midterm is in class on Fri. Oct. 10.
A midterm study aid and old exams are available from the class web page.

# Duality Theory: Finding an upper bound on the optimal solution.

Upper bound:

Function $g(x)$ is an upper bound for $f(x)$ if $f(x) \leq g(x)$.

Consider this problem:

Maximize $1\ x_1 + 6x_2 + 8x_3$

subject to

$$1\ x_1 + 3\ x_2 + 5\ x_3 \leq 5$$
$$0\ x_1 + 4\ x_2 + 3\ x_3 \leq 3$$
$$2\ x_1 + 0\ x_2 + 2\ x_3 \leq 2$$

$$x_1,\ x_2,\ x_3 \geq 0$$

Objective function: $x_1 + 6x_2 + 8x_3$

Constraints:

$$1\ x_1 + 3\ x_2 + 5\ x_3 \leq 5$$
$$0\ x_1 + 4\ x_2 + 3\ x_3 \leq 3$$
$$2\ x_1 + 0\ x_2 + 2\ x_3 \leq 2$$

Take 2 times the first constraint:

$$2*(1\ x_1 + 3\ x_2 + 5\ x_3) \leq 2*5$$

Hence, for all solutions:

$$2\ x_1 + 6\ x_2 + 10\ x_3 \leq 10$$

Objective function:

$1\ x_1 + 6\ x_2 + 8\ x_3$

For all solutions:

$2\ x_1 + 6\ x_2 + 10\ x_3 \leq\ 10$

Notice that since $x_1,\ x_2,\ x_3 \geq 0$

$1\ x_1 + 6\ x_2 + 8\ x_3$

$\leq$

$2\ x_1 + 6\ x_2 + 10\ x_3$

Objective function:

$1\ x_1 + 6\ x_2 + 8\ x_3$

For all solutions:

$2\ x_1 + 6\ x_2 + 10\ x_3 \leq\ 10$

Notice that since $x_1,\ x_2,\ x_3 \geq 0$

$1\ x_1 + 6\ x_2 + \ 8\ \ x_3$

$\leq$

$2\ x_1 + 6\ x_2 + 10\ x_3 \leq 10$

Therefore 10 is an upper bound on the optimal solution.

Objective function: $x_1 + 6x_2 + 8x_3$

Constraints:

$1\ x_1 + 3\ x_2 + 5\ x_3 \le 5$

$0\ x_1 + 4\ x_2 + 3\ x_3 \le 3$

$2\ x_1 + 0\ x_2 + 2\ x_3 \le 2$

Can we find a better upper bound?

Take 2 times the second constraint plus the third constraint:

$0\ x_1 + 8\ x_2 + 6\ x_3\ +$

$2\ x_1 + 0\ x_2 + 2\ x_3 \le 2*3+2$

Hence: $2\ x_1 + 8\ x_2 + 8\ x_3 \le 8$

Objective function:

$1\ x_1 + 6\ x_2 + 8\ x_3$

For all solutions:

$2\ x_1 + 8\ x_2 + 8\ x_3 \leq\ 8$

Notice that since $x_1,\ x_2,\ x_3 \geq 0$

$1\ x_1 + 6\ x_2 + 8\ x_3$

$\leq$

$2\ x_1 + 8\ x_2 + 8\ x_3 \leq\ 8$

So 8 is a tighter (better) upper bound on the solution (before we had 10).

Maximize $1\,x_1 + 6x_2 + 8x_3$
subject to

$$1\,x_1 + 3\,x_2 + 5\,x_3 \leq 5$$
$$0\,x_1 + 4\,x_2 + 3\,x_3 \leq 3$$
$$2\,x_1 + 0\,x_2 + 2\,x_3 \leq 2$$

$$x_1,\ x_2,\ x_3 \geq 0$$

Consider $x_1 = 0$, $x_2 = 0$, $x_3 = 1$.
What can we conclude?

Maximize $1\ x_1 + 6x_2 + 8x_3$
subject to
$1\ x_1 + 3\ x_2 + 5\ x_3 \leq 5$
$0\ x_1 + 4\ x_2 + 3\ x_3 \leq 3$
$2\ x_1 + 0\ x_2 + 2\ x_3 \leq 2$

Consider $x_1 = 0$, $x_2 = 0$, $x_3 = 1$.

This solution is feasible.
It has $z = 8$ and therefore $z \geq 8$.

So far:
We have concluded that $z \geq 8$ (because we have a feasible solution with $z=8$). So 8 is a lower bound on the optimal value for z.
We also concluded that $z \leq 8$ from considering a linear combination of the last two constraints.
Therefore, since our lower bound is equal to our upper bound, we are done!

The approach I was taking to get an upper bound was ad hoc (I just guessed something and got lucky).

How can we make this more systematic?


http://weheartit.com/tag/lucky.%20minion

I was choosing linear combinations of the constraints and trying to find combinations that dominate the objective function.

The goal was to try to minimize the upper bound we get for the objective function.

Another LP problem:

Maximize $x_1 + 2\,x_2$

subject to

$$
\begin{array}{rcrl}
x_1 & + & x_2 & \leq 10 \qquad \text{Take } y_1 \text{ times this.} \\
-2\,x_1 & + & x_2 & \leq 4 \qquad \text{Take } y_2 \text{ times this.}
\end{array}
$$

$x_1, x_2 \geq 0$

The objective function: $x_1 + 2\,x_2$

The constraints:

$$x_1 + x_2 \leq 10$$ Take $y_1$ times this.

$$-2\,x_1 + x_2 \leq 4$$ Take $y_2$ times this.

$$y_1\,x_1 + y_1\,x_2 \leq 10\,y_1$$

$$-2\,y_2\,x_1 + y_2\,x_2 \leq 4\,y_2$$

IMPORTANT: $y_1, y_2 \geq 0$ (otherwise the inequality sign flips)

Add them together:

$$(y_1 - 2y_2)\,x_1 + (y_1 + y_2)\,x_2 \leq 10\,y_1 + 4y_2$$

The objective function: $1\ x_1 + 2\ x_2$

The linear combination of the constraints:

$(y_1 - 2y_2)\ x_1 + (y_1 + y_2)\ x_2 \leq 10\ y_1 + 4y_2$

To dominate the objective function:

The coefficient of $x_1$: $y_1 - 2\ y_2 \geq 1$

The coefficient of $x_2$: $y_1 + y_2 \geq 2$

To minimize the upper bound we get:

Minimize $10\ y_1 + 4y_2$

The dual problem:

Minimize $10\ y_1 + 4y_2$ (upper bound)

subject to:

To dominate the objective function:

The coefficient of $x_1$: $y_1 - 2\ y_2 \geq 1$

The coefficient of $x_2$: $y_1 + y_2 \geq 2$

So the inequality signs do not flip:

$y_1\ ,\ y_2 \geq 0$

Put the dual problem into standard form:

Minimize $10\ y_1 + 4y_2$

$$y_1 - 2\ y_2 \geq 1$$
$$y_1 + y_2 \geq 2$$

$y_1\ , y_2 \geq 0$

What is the dual of the dual problem?

The primal:

Maximize $x_1 + 2\,x_2$

subject to

$$x_1 + x_2 \le 10$$
$$-2\,x_1 + x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Consider:

$x_1 = 2, \quad x_2 = 8$

The dual:

Minimize $10\,y_1 + 4y_2$

subject to:

$$y_1 - 2\,y_2 \ge 1$$
$$y_1 + y_2 \ge 2$$

$$y_1, y_2 \ge 0$$

Consider:

$y_1 = 5/3 \quad y_2 = 1/3$

What conclusions can we make?