In a previous lecture, this dictionary resulted after the first phase 1 pivot:

```
X0 = 3 - 1 X1 - 1 X2 + 1 X3
X4 = 5 - 4 X1 + 3 X2 + 1 X3
------------------------------
z = -3 + 1 X1 + 1 X2 - 1 X3
```

How can you solve a problem like this using your computer program?

1. What should you type in as input?

2. How should you handle the results when the program finishes?

If you still are having problems getting your program to work, send me e-mail for help. You will need to use it later in the term to start Programming Project #2 and to find answers to problems.

Assignment 2: Due at beginning of class on Fri. Oct. 3. Late submissions accepted with 10% penalty until Tues. Oct. 7 at 12:30pm.

Our midterm is in class on Fri. Oct. 10.
A midterm study aid and old exams are available from the class web page.

# Last class:

The primal:
Maximize $x_1 + 2x_2$
subject to

$$x_1 + x_2 \leq 10$$
$$-2x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

The dual:
Minimize $10y_1 + 4y_2$
subject to:

$$y_1 - 2y_2 \geq 1$$
$$y_1 + y_2 \geq 2$$

$$y_1, y_2 \geq 0$$

The primal:

Maximize $x_1 + 2x_2$

subject to

$$\begin{aligned} x_1 + x_2 &\le 10 \implies x_1 + x_2 + x_3 = 10 \\ -2x_1 + x_2 &\le 4 \implies -2x_1 + x_2 + x_4 = 4 \end{aligned}$$

We can see the multiple of each equation that was subtracted from the z row by the coefficient of its slack variable.

Consequence:
The dual solution is hiding in the final dictionary for the primal problem:

$y_i$= -1 * (coefficient in the z row of the ith slack variable).

For the primal:

The initial dictionary:

```
X3 = 10.00-  1.00 X1 -  1.00 X2
X4 =  4.00+  2.00 X1 -  1.00 X2
-------------------------------
z  =  0.00+  1.00 X1 +  2.00 X2
```

After  2 pivots:

```
X1 =  2.00-  0.33 X3 +  0.33 X4
X2 =  8.00-  0.67 X3 -  0.33 X4
-------------------------------
z  = 18.00-  1.67 X3 -  0.33 X4
```

The dual:

Minimize $10\ y_1 + 4y_2$

subject to:

$$y_1 - 2\,y_2 \geq 1$$

$$y_1 + y_2 \geq 2$$

$$y_1,\ y_2 \geq 0$$

Get the first initial feasible dictionary for Phase 1 for this problem.

For the dual:
The initial dictionary:
Phase 1: Before pivoting to make feasible.

```
Y3 = -1.0+ 1.0 Y1 -2.0 Y2 +1.0 Y0
Y4 = -2.0+ 1.0 Y1 +1.0 Y2 +1.0 Y0
---------------------------------
z =    0.0+ 0.0 Y1 +0.0 Y2 -1.0 Y0
```

Where should the first pivot be to make this feasible?

Y0 enters.  Y4 leaves.
The initial dictionary:

```
Y3 = 1.0+ 0.0 Y1 -3.0 Y2 +1.0 Y4
Y0 = 2.0- 1.0 Y1 -1.0 Y2 +1.0 Y4
--------------------------------
z = -2.0+ 1.0 Y1 +1.0 Y2 -1.0 Y4
```

Y1 enters.  Y0 leaves. z = -2.0

The last Phase 1 Dictionary:

```
Y3 = 1.0- 3.0 Y2 +1.0 Y4 +0.0 Y0
Y1 = 2.0- 1.0 Y2 +1.0 Y4 -1.0 Y0
---------------------------------
z = 0.0+ 0.0 Y2  +0.0 Y4 -1.0 Y0
```

The objective function:

Minimize 10 $y_1$ +4$y_2$

What dictionary do we use to start Phase 2?

Be careful: standard form is to MAXIMIZE.

The first Phase Dictionary:

```
Y3 =  1.00-  3.00 Y2 +  1.00 Y4
Y1 =  2.00-  1.00 Y2 +  1.00 Y4
--------------------------------
z  =-20.00+  6.00 Y2 - 10.00 Y4
```

Y2 enters.Y3 leaves. z = -20.0

After 1 pivot:

```
Y2 =  0.33-  0.33 Y3 +  0.33 Y4
Y1 =  1.67+  0.33 Y3 +  0.67 Y4
-------------------------------
z  =-18.00-  2.00 Y3 -  8.00 Y4
```

The dual solution is hiding in the final dictionary for the primal problem:

$y_i = -1 * $ (coeff. in z row of the ith slack variable).

Since the dual of the dual is the primal, the primal solution is hiding in the final dictionary for the dual problem.

$x_i = -1 * $ (coeff. in z row of the ith slack variable).

The final dual dictionary:

```
Y2 =  0.33-  0.33 Y3 +  0.33 Y4
Y1 =  1.67+  0.33 Y3 +  0.67 Y4
------------------------------
z  =-18.00-  2.00 Y3 -  8.00 Y4
```

The primal:

Maximize $x_1 + 2\,x_2$

subject to

$$x_1 + x_2 \le 10$$

$$-2\,x_1 + x_2 \le 4$$

$$x_1,\ x_2 \ge 0$$

The primal:

Maximize $x_1 + 2x_2$

subject to

$$x_1 + x_2 \le 10$$
$$-2x_1 + x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Consider:

$x_1 = 2, \quad x_2 = 8$

The dual:

Minimize $10y_1 + 4y_2$

subject to:

$$y_1 - 2y_2 \ge 1$$
$$y_1 + y_2 \ge 2$$

$$y_1, y_2 \ge 0$$

Consider:

$y_1 = 5/3 \quad y_2 = 1/3$

What conclusions can we make?

Maximize $2\,x_1 - 1\,x_2 + 3\,x_3$

subject to

$$
\begin{aligned}
2\,x_1 &+ 4\,x_2 - 1\,x_3 \le 3 \\
1\,x_1 &- 1\,x_2 + 0\,x_3 \le 5 \\
2\,x_1 &+ 1\,x_2 + 1\,x_3 \le 2
\end{aligned}
$$

$x_1,\ x_2,\ x_3 \ge 0$

1. What is the dual?
2. Put the dual in standard form.

$$\text{Maximize } 2x_1 - 1x_2 + 3x_3$$

subject to

$$2x_1 + 4x_2 - 1x_3 \le 3$$
$$1x_1 - 1x_2 + 0x_3 \le 5$$
$$2x_1 + 1x_2 + 1x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

Question 1: What is the dual?

$$\text{Minimize } 3y_1 + 5y_2 + 2y_3$$

subject to

$$2y_1 + 1y_2 + 2y_3 \ge 2$$
$$4y_1 - 1y_2 + 1y_3 \ge -1$$
$$-1y_1 + 0y_2 + 1y_3 \ge 3$$
$$y_1, y_2, y_3 \ge 0$$

Minimize $3y_1 + 5y_2 + 2y_3$

subject to

$$\begin{aligned} 2y_1 + 1y_2 + 2y_3 &\geq 2 \\ 4y_1 - 1y_2 + 1y_3 &\geq -1 \\ -1y_1 + 0y_2 + 1y_3 &\geq 3 \end{aligned}$$

$y_1, y_2, y_3 \geq 0$

Solution 2: Standard form.

Maximize $-3y_1 - 5y_2 - 2y_3$

subject to

$$\begin{aligned} -2y_1 - 1y_2 - 2y_3 &\leq -2 \\ -4y_1 + 1y_2 - 1y_3 &\leq 1 \\ 1y_1 - 0y_2 - 1y_3 &\leq -3 \end{aligned}$$

$y_1, y_2, y_3 \geq 0$

Numerical analysis practitioners usually us x to denote a column vector and $x^T$ for a row vector.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_k \end{pmatrix} \qquad x^T = (x_1 \ x_2 \ x_3 \ \ldots \ x_k)$$

This is the convention I will use. Note that it sometimes differs from our text.

## Matrix notation for duality:

The primal:

Maximize $c^T x$

subject to

$Ax \leq b$,

$x \geq 0$.

The dual:

Minimize $b^T y$

subject to

$A^T y \geq c$

$y \geq 0$.

The dual in standard form:

Maximize $-b^T y$

subject to

$-A^T y \leq -c$,

$y \geq 0$.

Question 3: Solve the primal and dual problems using the Simplex method.

Solution 3: Solve the primal.

```
X4 =   3 - 2 X1 - 4 X2 + 1 X3
X5 =   5 - 1 X1 + 1 X2 + 0 X3
X6 =   2 - 2 X1 - 1 X2 - 1 X3
------------------------------
z  =   0 + 2 X1 - 1 X2 + 3 X3
```

X3 enters. X6 leaves. z = 0

After 1 pivot:

```
X4 =   5 - 4 X1 - 5 X2 - 1 X6
X5 =   5 - 1 X1 + 1 X2 + 0 X6
X3 =   2 - 2 X1 - 1 X2 - 1 X6
-----------------------------
z  =   6 - 4 X1 - 4 X2 - 3 X6
```

The optimal solution: 6

Solution 3: Solve the dual.

Maximize $-3\ y_1 \quad -5\ y_2 \quad -2\ y_3$

subject to

$$
\begin{array}{rrrr}
-2\ y_1 & -\ 1\ y_2 & -\ 2\ y_3 & \leq -2 \\
-4\ y_1 & +\ 1\ y_2 & -\ 1\ y_3 & \leq\ 1 \\
1\ y_1 & -\ 0\ y_2 & -\ 1\ y_3 & -3
\end{array}
$$

$y_1,\ y_2,\ y_3 \geq 0$

Phase 1: Which variable should leave?

```
Y4 =  -2 + 2 Y1 + 1 Y2 + 2 Y3 + 1 Y0
Y5 =   1 + 4 Y1 - 1 Y2 + 1 Y3 + 1 Y0
Y6 =  -3 - 1 Y1 + 0 Y2 + 1 Y3 + 1 Y0
-----------------------------------
z  =  0 + 0 Y1 + 0 Y2 + 0 Y3 - 1 Y0
```

The initial feasible dictionary for phase 1:

```
Y4 =   1 + 3 Y1 + 1 Y2 + 1 Y3 + 1 Y6
Y5 =   4 + 5 Y1 - 1 Y2 + 0 Y3 + 1 Y6
Y0 =   3 + 1 Y1 + 0 Y2 - 1 Y3 + 1 Y6
---------------------------------
z  =  -3 - 1 Y1 + 0 Y2 + 1 Y3 - 1 Y6
```

Y3 enters.   Y0 leaves.  z = -3

```
After   1 pivot:
Y4 =   4 + 4 Y1 + 1 Y2 + 2 Y6 - 1 Y0
Y5 =   4 + 5 Y1 - 1 Y2 + 1 Y6 + 0 Y0
Y3 =   3 + 1 Y1 + 0 Y2 + 1 Y6 - 1 Y0
-----------------------------------
z  =   0 + 0 Y1 + 0 Y2 + 0 Y6 - 1 Y0
```

The optimal solution: 0

What dictionary should be used to start phase 2?

The objective function:
Maximize $-3\ y_1\ \ -5\ y_2\ \ -2\ y_3$

The initial dictionary:

```
Y4 =    4 + 4 Y1 + 1 Y2 + 2 Y6
Y5 =    4 + 5 Y1 - 1 Y2 + 1 Y6
Y3 =    3 + 1 Y1 + 0 Y2 + 1 Y6
------------------------------
z  =   -6 - 5 Y1 - 5 Y2 - 2 Y6
```

The dual   optimal solution: -6
The primal optimal solution:  6

Why are they not equal?

Question 4: Explain how to get the primal solution from the dual and the dual solution from the primal.

The final primal dictionary:

```
X4 =   5 - 4 X1 - 5 X2 - 1 X6
X5 =   5 - 1 X1 + 1 X2 + 0 X6
X3 =   2 - 2 X1 - 1 X2 - 1 X6
------------------------------
z  =   6 - 4 X1 - 4 X2 - 3 X6
```

Look at the z row of final dictionary:

```
Y1 = -1 * (coeff. of 1st slack, X4)=  0
Y2 = -1 * (coeff. of 2nd slack, X5)=  0
Y3 = -1 * (coeff. of 3rd slack, X6)=  3
```

The final dual dictionary:

```
Y4 =   4 + 4 Y1 + 1 Y2 + 2 Y6
Y5 =   4 + 5 Y1 - 1 Y2 + 1 Y6
Y3 =   3 + 1 Y1 + 0 Y2 + 1 Y6
------------------------------
z  =  -6 - 5 Y1 - 5 Y2 - 2 Y6
```

Look at the z row of final dictionary:

```
X1 = -1 *(coeff. of 1st slack, Y4)= 0
X2 = -1 *(coeff. of 2nd slack, Y5)= 0
X3 = -1 *(coeff. of 3rd slack, Y6)= 2
```

Maximize $2\ x_1 - 1\ x_2 + 3\ x_3$

subject to

PRIMAL

$$2\ x_1 + 4\ x_2 - 1\ x_3 \leq 3$$

$$1\ x_1 - 1\ x_2 + 0\ x_3 \leq 5$$

$$2\ x_1 + 1\ x_2 + 1\ x_3 \leq 2$$

$x_1,\ x_2,\ x_3 \geq 0$ $\quad x = (0,0,2)$

==========================

Minimize $3\ y_1 + 5\ y_2 + 2\ y_3$

subject to

DUAL

$$2\ y_1 + 1\ y_2 + 2\ y_3 \geq 2$$

$$4\ y_1 - 1\ y_2 + 1\ y_3 \geq -1$$

$$-1\ y_1 + 0\ y_2 + 1\ y_3 \geq 3$$

$y_1,\ y_2,\ y_3 \geq 0$ $\quad y = (0,0,3)$

The primal solution is: (0, 0, 2)
The dual solution is: (0, 0, 3)
We know that both of these are optimal solutions because:

1. (0, 0, 2) is primal feasible.

2. (0, 0, 3) is dual feasible.

3. The value of the primal at (0,0,2) equals the value of the dual at (0, 0, 3) [both are equal to 6].

The primal:

Maximize $c^T x$

subject to

$Ax \leq b$,

$x \geq 0$.

The dual:

Minimize $b^T y$

subject to

$A^T y \geq c$

$y \geq 0$.

**The Duality Theorem**

If the primal has an optimal solution $x^*$ with $z = c^T x^*$, then the dual also has an optimal solution $y^*$, and $b^T y^* = c^T x^*$.

A linear programming problem can have an optimal solution, or it can be infeasible or unbounded.

Thought question:
which combinations are possible?

| Primal\Dual | Optimal | Infeasible | Unbounded |
|---|---|---|---|
| Optimal | | | |
| Infeasible | | | |
| Unbounded | | | |