Bob and Sue solved this by hand:

Maximize $x_1 + 2\ x_2$ subject to

$1\ x_1 + 1\ x_2 \leq 10$

$-2\ x_1 + 1\ x_2 \leq 4$

$x_1,\ x_2 \geq 0$

and their last dictionary was:

```
X1 = 10.00-  1.00 X2 -  1.00 X3
X4 = 24.00-  3.00 X2 -  2.00 X3
--------------------------------
z  = 10.00 -  1.00 X2 -  1.00 X3
```

1. What is the dual?
2. Use the dictionary to find y_1 and y_2.
3. Use duality theory to see if this is a correct solution to this problem.

Bob and Sue solved this by hand:

Maximize $x_1 + 2\ x_2$ subject to

$1\ x_1 + 1\ x_2 \leq 10$

$-2\ x_1 + 1\ x_2 \leq 4$

$x_1,\ x_2 \geq 0$

corrected solution:

```
X1 = 10.00-  1.00 X2 -  1.00 X3
X4 = 24.00-  3.00 X2 -  2.00 X3
--------------------------------
z  = 10.00 +  1.00 X2 -  1.00 X3
```

After 2 pivots:

```
X1 =  2.00-  0.33 X3 +  0.33 X4
X2 =  8.00-  0.67 X3 -  0.33 X4
--------------------------------
z  = 18.00-  1.67 X3 -  0.33 X4
```

If you still are having problems getting your program to work, send me e-mail for help

Assignment 2: Due at beginning of class on Fri. Oct. 3. Late submissions accepted with 10% penalty until Tues. Oct. 7 at 12:30pm.

Our midterm is in class on Fri. Oct. 10.
A midterm study aid and old exams are available from the class web page.
Tutorial: Thurs. Oct. 9, 6pm, ECS 108.

Digression today: Slides 18 onwards of Lecture 11.

The primal: Maximize $c^T x$
subject to $A x \leq b$, $x \geq 0$.

The dual: Minimize $b^T y$
subject to $A^T y \geq c$, $y \geq 0$.

The Duality Theorem

If the primal has an optimal solution x^* with $z = c^T x^*$, then the dual also has an optimal solution y^*, and $b^T y^* = c^T x^*$.

Consider

Maximize $4 X_1 + 1 X_2 + 5 X_3 + 3 X_4$

subject to

$$
\begin{aligned}
1 X_1 - 1 X_2 - 1 X_3 + 3 X_4 &\leq 1 \\
5 X_1 + 1 X_2 + 3 X_3 + 8 X_4 &\leq 55 \\
-1 X_1 + 2 X_2 + 3 X_3 - 5 X_4 &\leq 3
\end{aligned}
$$

$X_1, X_2, X_3, X_4 \geq 0$

The initial dictionary:

```
X5 =  1-  1 X1 +  1 X2 +  1 X3 -  3 X4
X6 = 55-  5 X1 -  1 X2 -  3 X3 -  8 X4
X7 =  3+  1 X1 -  2 X2 -  3 X3 +  5 X4
----------------------------------------
z = -0+  4 X1 +  1 X2 +  5 X3 +  3 X4
```

After 3 pivots:

```
X4 =  5-  1 X1 -  1 X3 -  2 X5 -  1 X7
X6 =  1+  5 X1 +  9 X3 + 21 X5 + 11 X7
X2 = 14-  2 X1 -  4 X3 -  5 X5 -  3 X7
---------------------------------------
z =  29-  1 X1 -  2 X3 - 11 X5 -  6 X7
```

The optimal solution: 28.999992 (actually 29).

After 3 pivots:

```
X4 =   5- 1 X1 - 1 X3 - 2 X5 - 1 X7
X6 =   1+ 5 X1 + 9 X3 +21 X5 +11 X7
X2 =  14- 2 X1 - 4 X3 - 5 X5 - 3 X7
---------------------------------
z  =  29- 1 X1 - 2 X3 -11 X5 - 6 X7
```

Take y_1= -(coeff. of X5 in Z row)=11

Take y_2= -(coeff. of X6 in Z row)= 0

Take y_3= -(coeff. of X7 in Z row)= 6

In general, set y_i= -1 * (coeff. of ith slack variable in the Z row).

The initial dictionary:

```
X5 =  1-  1 X1 +  1 X2 +  1 X3 -  3 X4
X6 = 55-  5 X1 -  1 X2 -  3 X3 -  8 X4
X7 =  3+  1 X1 -  2 X2 -  3 X3 +  5 X4
---------------------------------------
z = -0+  4 X1 +  1 X2 +  5 X3 +  3 X4
```

Multiply the 3 rows of the initial dictionary by y_1, y_2, and y_3 respectively:

```
11 X5 = 11 -11 X1 +11 X2 +11 X3 -33 X4
 0 X6 =  0 - 0 X1 - 0 X2 - 0 X3 - 0 X4
 6 X7 = 18 + 6 X1 -12 X2 -18 X3 +30 X4
```

Multiply the 3 rows of the initial dictionary by y_1, y_2, and y_3 respectively then add them together:

```
11 X5 = 11 -11 X1 +11 X2 +11 X3 -33 X4
 0 X6 =  0 - 0 X1 - 0 X2 - 0 X3 - 0 X4
 6 X7 = 18 + 6 X1 -12 X2 -18 X3 +30 X4
=======================================
11 X5 +6 X7
      = 29  -5 X1 - 1 X2 - 7 X3 - 3 X4
```

Rearrange:

```
0= 29 -5 X1  - 1 X2 - 7 X3 - 3 X4
    - 11 X5  - 6 X7
```

Rearrange:

```
0= 29 -5 X1   - 1 X2 - 7 X3 - 3 X4
     - 11 X5   - 6 X7
```

Add to the original equation for z:

```
z=  0 +4 X1 +1 X2 +5 X3 +3 X4
0= 29 -5 X1 -1 X2 -7 X3 -3 X4 -11 X5 -6 X7
-------------------------------------------
z= 29- 1 X1 +0 X2 -2 X3 -0 X4 -11 X5 -6 X7
```

```
z=  0 +4 X1 +1 X2 +5 X3 +3 X4
0= 29 -5 X1 -1 X2 -7 X3 -3 X4 -11 X5 -6 X7
------------------------------------------
z= 29- 1 X1 +0 X2 -2 X3 -0 X4 -11 X5 -6 X7
```

The last dictionary:

```
X4 =   5- 1 X1 - 1 X3 - 2 X5 - 1 X7
X6 =   1+ 5 X1 + 9 X3 +21 X5 +11 X7
X2 =  14- 2 X1 - 4 X3 - 5 X5 - 3 X7
--------------------------------------
z  =  29- 1 X1 - 2 X3 -11 X5 - 6 X7
```

It is no surprise that this is the same as the z row in the final tableau. The Simplex method proceeds by adding linear combinations of the rows to the z row. The linear combination that led to the final result can be determined by looking at the coefficients of the slack variables because each slack variable is in only one of the original equations, and its original coefficient is 1.

Why in general must y as determined this way be dual feasible?

First, $y \geq 0$ because if the value of y_i was negative the corresponding coefficient in the z row would be strictly positive, and hence the Simplex method would continue to pivot.

Second, The fact that these constant multiples of the original equations yield coefficients of the x_i's that dominate those in the objective function is an artifact of the fact that we do not stop pivoting until the coefficients of x_i's in the z row are negative or zero.

The dual: Minimize $b^T y$
subject to $A^T y \geq c$, $y \geq 0$.

Writing this out:

Minimize $b_1 y_1 + b_2 y_2 + \ldots. + b_m y_m$
subject to

$$a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m \geq c_1$$
$$a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m \geq c_2$$
$$\ldots$$
$$a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m \geq c_m$$

$$y_1, y_2, \ldots, y_m \geq 0$$

DUAL CONSTRAINT:

$$a_{11}\ y_1 + a_{21}\ y_2 + \ldots + a_{m1}\ y_m \geq c_1$$

The z row when we are done has the coefficient of x_1 equal to:

$$c_1 - (a_{11}\ y_1 + a_{21}\ y_2 + \ldots + a_{m1}\ y_m)$$

The coefficient of x_1 must be ≤ 0 since otherwise we would keep pivoting.

This happens exactly when:

$$a_{11}\ y_1 + a_{21}\ y_2 + \ldots + a_{m1}\ y_m \geq c_1$$

So this dual constraint is satisfied.

Finally, the objective function values are the same for the dual as for the primal because the values are just

$$b_1 \; y_1 \;\; + \; b_2 \; y_2 \; + \; .... \; + \; b_m \; y_m$$

The primal value of z equals this because this because we added this linear combination of the b_i's to the z row, and the dual value equals this by definition.

A linear programming problem can have an optimal solution, or it can be infeasible or unbounded.

Thought question:
which combinations are possible?

Primal\Dual	Optimal	Infeasible	Unbounded
Optimal			
Infeasible			
Unbounded			