Use duality theory to check if the solution implied by the dictionary is correct.

Maximize $x_1 + 3\ x_2 + 6\ x_3$

subject to

$1\ x_1 + 1\ x_2 - 1\ x_3 \leq 5$

$1\ x_1 + 0\ x_2 + 2\ x_3 \leq 3$

$0\ x_1 + 1\ x_2 + 6\ x_3 \leq 4$

$x_1, x_2, x_3 \geq 0$

```
X2 =  2+  3 X3 -  1 X4 +  1 X6
X1 =  3-  2 X3 +  0 X4 -  1 X6
X5 =  2-  9 X3 +  1 X4 -  1 X6
-------------------------------
z  =  9- 13 X3 -  1 X4 -  1 X6
```

Maximize $x_1 + 3 x_2 + 6 x_3$
subject to
$1 x_1 + 1 x_2 - 1 x_3 \leq 5$
$1 x_1 + 0 x_2 + 2 x_3 \leq 3$
$0 x_1 + 1 x_2 + 6 x_3 \leq 4$
$x_1, x_2, x_3 \geq 0$

After 4 pivots:
```
X2 =  4 -  6 X3 +  0 X4 -  1 X6
X1 =  1 +  7 X3 -  1 X4 +  1 X6
X5 =  2 -  9 X3 +  1 X4 -  1 X6
-------------------------------
z  = 13 -  5 X3 -  1 X4 -  2 X6
```

If you still are having problems getting your program to work, send me e-mail for help

Assignment 2: Due at beginning of class on Fri. Oct. 3. Late submissions accepted with 10% penalty until Tues. Oct. 7 at 12:30pm.

Our midterm is in class on Fri. Oct. 10.
A midterm study aid and old exams are available from the class web page.
Tutorial: Thurs. Oct. 9, 6pm, ECS 108.

The primal: Maximize $c^T x$
subject to $A x \le b$, $x \ge 0$.

Writing this out:

Maximize $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$
subject to

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \le b_1$$
$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \le b_2$$
$$\dots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \le b_m$$

$$x_1, x_2, \dots, x_n \ge 0$$

The dual: Minimize $b^T y$
subject to $A^T y \geq c$, $y \geq 0$.

Writing this out:

Minimize $b_1 y_1 + b_2 y_2 + \ldots. + b_m y_m$
subject to

$a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m \geq c_1$
$a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m \geq c_2$
...
$a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m \geq c_m$

$y_1, y_2, \ldots, y_m \geq 0$

Theorem: For every primal feasible solution $x = (x_1, x_2, \ldots, x_n)$ and for every dual feasible solution $y = (y_1, y_2, \ldots, y_m)$,

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \leq$$

$$b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

That is, the objective function values for dual feasible solutions always give upper bounds on the objective function values for primal feasible solutions.

Proof: Because the constraints of the dual are (and $x \geq 0$):

$$a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m \geq c_1$$
$$a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m \geq c_2$$
$$\ldots$$
$$a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m \geq c_n$$

we have that S=

$$(a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m) x_1 +$$
$$(a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m) x_2 +$$
$$\ldots$$
$$(a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m) x_n$$

$$\geq c_1 x_1 + c_2 x_2 + \ldots\ldots + c_n x_n$$

We have that S=

$(a_{11}\ y_1 + a_{21}\ y_2 + \ ...\ + a_{m1}\ y_m) x_1 +$
$(a_{12}\ y_1 + a_{22}\ y_2 + \ ...\ + a_{m2}\ y_m) x_2 +$
$...\ +$
$(a_{1n}\ y_1 + a_{2n}\ y_2 + \ ...\ + a_{mn}\ y_m)\ x_n$
$\geq c_1\ x_1 + c_2\ x_2 + \ ....\ + c_n\ x_n$

Regroup the terms on the LHS so they are in terms of x_i's instead of y_j's:

S=

$(a_{11}\ x_1 + a_{12}\ x_2 + \ ...\ + a_{1n}\ x_n) y_1 +$
$(a_{21}\ x_1 + a_{22}\ x_2 + \ ...\ + a_{2n}\ x_n) y_2 +$
$...\ +$
$(a_{m1}\ x_1 + a_{m2}\ x_2 + \ ...\ + a_{mn}\ x_n)\ y_m$

$S =$

$(a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n) y_1 +$
$(a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n) y_2 +$
$\dots +$
$(a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n) y_m$

From the primal problem:

$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq b_1$
$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq b_2$
$\dots$
$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq b_m$

Therefore since $y \geq 0$,

$S \leq$
$b_1 y_1 + b_2 y_2 + \dots + b_m y_m$

Summing it up, we have shown that

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$
$$\leq S \leq$$
$$b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

so by transitivity, this proves what we were trying to show:

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \leq b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

Important observation: it is critical for preserving the direction of the inequalities that the solutions x and y are feasible and hence non-negative.

The primal: Maximize $c^T x$
subject to $A x \le b$, $x \ge 0$.

The dual: Minimize $b^T y$
subject to $A^T y \ge c$, $y \ge 0$.

The Duality Theorem

If the primal has an optimal solution x^* with $z = c^T x^*$, then the dual also has an optimal solution y^*, and $b^T y^* = c^T x^*$.

A linear programming problem can have an optimal solution, or it can be infeasible or unbounded.

Thought question:
which combinations are possible?

Primal\Dual	Optimal	Infeasible	Unbounded
Optimal			
Infeasible			
Unbounded			

The Duality Theorem

If the primal has an optimal solution x^* with $z = c^T x^*$, then the dual also has an optimal solution y^*, and $b^T y^* = c^T x^*$.

Primal\Dual	Optimal	Infeasible	Unbounded
Optimal	YES	NO	NO
Infeasible	NO		
Unbounded	NO		

Theorem (last class): For every primal feasible solution $x = (x_1, x_2, \ldots, x_n)$ and for every dual feasible solution $y = (y_1, y_2, \ldots, y_m)$,

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \leq b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

Primal\Dual	Optimal	Infeasible	Unbounded
Optimal	YES	NO	NO
Infeasible	NO	?	YES
Unbounded	NO	YES	NO

Example on assignment that is due today:

Primal\Dual	Optimal	Infeasible	Unbounded
Optimal	YES	NO	NO
Infeasible	NO	YES	YES
Unbounded	NO	YES	NO

Another example:

Maximize $2 X1 - X2$

subject to

$$X1 - X2 \leq 1$$

$$-X1 + X2 \leq -2$$

$$X \geq 0$$

What is the dual?

Draw a picture of this.

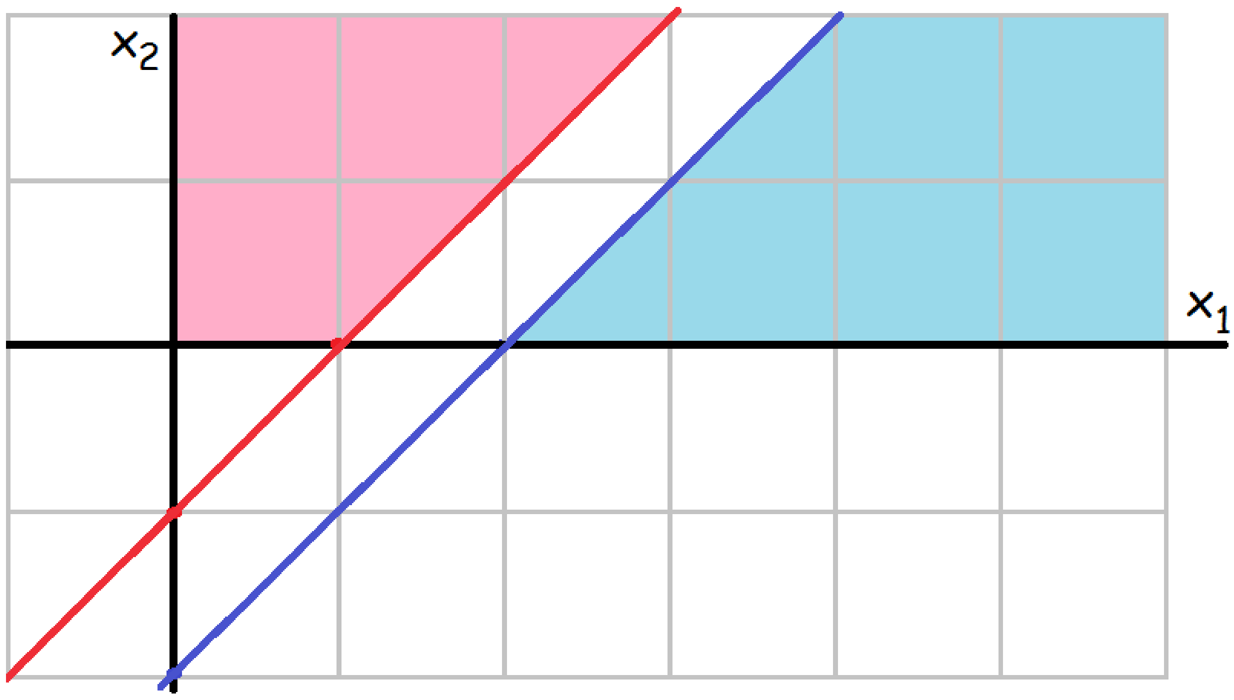
Maximize 2 X1 - X2
subject to

$X1 - X2 \leq 1$

$-X1 + X2 \leq -2$

$X \geq 0$

The Primal is not feasible.



Maximize $2 X_1 - X_2$
subject to

$X_1 - X_2 \leq 1$

$-X_1 + X_2 \leq -2$

$X \geq 0$

The dual is:

Minimize

$1 Y_1 - 2 Y_2$

subject to

$Y_1 - Y_2 \geq 2$

$-Y_1 + Y_2 \geq -1$

$Y \geq 0$

The dual in standard form:

Maximize

$- Y_1 + 2 Y_2$

subject to

$- Y_1 + Y_2 \leq -2$

$Y_1 - Y_2 \leq 1$

$Y \geq 0$

Maximize 2 X1 - X2
subject to

$$X1 - X2 \leq 1$$
$$-X1 + X2 \leq -2$$
$$X \geq 0$$

Last dictionary for phase 1:

$$X1 = 1.5 + 1\ X2 - 0.5\ X3 + 0.5\ X4$$
$$X0 = 0.5 + 0\ X2 + 0.5\ X3 + 0.5\ X4$$

$$z = -0.5 + 0\ X2 - 0.5\ X3 - 0.5\ X4$$

X1=1.5, X2=0, Z= -0.5

Maximize 2 X1 - X2

subject to

X1 - X2 ≤ 1

1.5 - 0 misses constraint by -0.5

-X1 + X2 ≤ -2

-1.5 + 0 misses constraint by -0.5

z tells us how badly the worst constraint is missing its constraint.

Last dictionary:

X1=1.5, X2=0, Z= -0.5