The initial dictionary:

```
X5= 30 -3 X1 -3 X2 +0 X3 +0 X4
X6= 16 +0 X1 +0 X2 -2 X3 -2 X4
X7= 12 -1 X1 -2 X2 -3 X3 -4 X4
X8= 24 -2 X1 -2 X2 -6 X3 -8 X4
-------------------------------
z =  0 +1 X1 +2 X2 +3 X3 +4 X4
```

The final dictionary:

```
X1 = a +3 X3 +4 X4 -2/3 X5 +1 X7
X6 = b -2 X3 -2 X4 + 0  X5 +0 X7
X2 = c -3 X3 -4 X4 +1/3 X5 -1 X7
X8 = d -6 X3 -8 X4 +2/3 X5 +0 X7
--------------------------------
z  = e +0 X3 +0 X4 + 0  X5 -1 X7
```

1. What matrix A in the computer corresponds to the initial dictionary?
2. What is the basis matrix B for the final dictionary?
3. What matrix A corresponds to the final dictionary?
4. How can you determine B^{-1} from the final dictionary?
5. Solve for a, b, c, d, and e.

Announcements

The midterm is in class on Friday Oct. 10.
A midterm study aid and some old midterms have been posted.

No office hours on Friday Oct. 10.

Special midterm tutorial: Thursday Oct. 9, 6pm, ECS 108. Bring any questions you have about anything in the class. I can go through solutions for assignment questions.

Student: Solves primal or dual problem whichever is easiest.

Primal: 9 variables, and 100 equations.
Dual: 100 variables, and 9 equations.

Text: claims that in practice, it is usually best to choose the problem with the least rows.

Student reads both primal and dual solutions from last dictionary.

Certificate: x^* and y^*, the optimal primal and dual solutions.

Supervisor: Has certificate plus primal and dual problems.

1. Check that x^* is primal feasible.

2. Check that y^* is dual feasible.

3. Check $c^T x^* = b^T y^*$.

These checks guarantee optimality.

Easier usually than solving LP problem.

The problems:

The primal problem:

Maximize

$- 3 X_1 - 2 X_2$

subject to

$1 X_1 -1 X_2 \leq -3$

$-2 X_1 -1 X_2 \leq -5$

$-0 X_1 -2 X_2 \leq -1$

$-1 X_1 -4 X_2 \leq 0$

$X_1, X_2 \geq 0$

The dual problem in standard form:

Maximize

$3Y_1 + 5Y_2 + 1Y_3 + 0Y_4$

subject to

$-1Y_1 +2Y_2 +0Y_3 +1Y_4 \leq 3$

$1Y_1 +1Y_2 +2Y_3 +4Y_4 \leq 2$

$Y_1, Y_2, Y_3, Y_4 \geq 0$

The student notices that the primal has no initial feasible basis so cannot be handled by the code written for the project. So the student solves the dual instead:

The initial dictionary:

```
Y5= 3 + 1 Y1 -2 Y2 + 0 Y3 -1 Y4
Y6= 2 - 1 Y1 -1 Y2 - 2 Y3 -4 Y4
--------------------------------
z = 0 + 3 Y1 +5 Y2 + 1 Y3 +0 Y4
```

After 2 pivots:

Y2= 1.67 -0.67 Y3 -1.67 Y4 -0.33 Y5 -0.33 Y6
Y1= 0.33 -1.33 Y3 -2.33 Y4 +0.33 Y5 -0.67 Y6
----
z = 9.33 -6.33 Y3 -15.33 Y4 -0.67 Y5 -3.67 Y6

The student guesses that 0.33 really means 1/3 and 0.67 is 2/3. So the certificate is:

Y_1= 1/3, Y_2= 5/3, Y_3= 0, Y_4= 0

X_1 = -1 (coeff. of Y5)= 2/3
X_2 = -1 (coeff. of Y6)= 3 2/3 = 11/3

The primal problem:

Maximize

$- 3 X_1 - 2 X_2$

subject to

$X_1 = 2/3$

$X_2 = 11/3$

Check for primal feasibility, the red equations are tight:

$1 X_1 - 1 X_2 \leq -3$	$1(2/3)-1(11/3) = -9/3 \leq -3$
$-2 X_1 - 1 X_2 \leq -5$	$-2(2/3)-1(11/3) = -15/3 \leq -5$
$-0 X_1 - 2 X_2 \leq -1$	$0(2/3)-2(11/3) = -22/3 \leq -1$
$-1 X_1 - 4 X_2 \leq 0$	$-1(2/3)-4(11/3) = -46/3 \leq 0$
$X_1, X_2 \geq 0$	$2/3, 11/3 \geq 0$

Check $Y_1 = 1/3$, $Y_2 = 5/3$, $Y_3 = 0$, $Y_4 = 0$ for dual feasibility:

$-1Y_1 + 2Y_2 + 0Y_3 + 1Y_4 \leq 3$
$1Y_1 + 1Y_2 + 2Y_3 + 4Y_4 \leq 2$
$Y_1, Y_2, Y_3, Y_4 \geq 0$

$-1(1/3) + 2(5/3) + 0*0 + 1*0 = 9/3 \leq 3$
$1(1/3) + 1(5/3) + 2*0 + 4*0 = 6/3 \leq 2$

$1/3, 5/3, 0, 0 \geq 0$

Check the objective function values:

The primal problem:
Maximize
$- 3 X_1 - 2 X_2$

$X_1 = 2/3$
$X_2 = 11/3$

$-6/3-22/3= -28/3$

The dual (not in standard form):
Minimize
$-3Y_1 -5Y_2 -1Y_3 + 0Y_4$

$Y_1= 1/3$, $Y_2= 5/3$,
$Y_3= 0$, $Y_4= 0$

$-1 -25/3 -0+ 0= -28/3$

Conclusion:

- (2/3, 11/3) is an optimal primal solution.
- (1/3, 5/3, 0, 0) an optimal dual solution.

Complementary slackness allows us to find a dual solution from a primal one.

What can we conclude by looking at the equations which are not tight?

The slacks are then non-zero and hence must be in the basis and so they have a (implicit) zero coefficient in the z row.

So the corresponding y variables must be 0.

Conclusion: if $x_{n+i} \neq 0$ then $y_i = 0$.

The red equations are tight:

			Slack
$1(2/3)-1(11/3) =$	$-9/3$	≤ -3	x_3
$-2(2/3)-1(11/3) =$	$-15/3$	≤ -5	x_4
$0(2/3)-2(11/3) =$	$-22/3$	≤ -1	x_5
$-1(2/3)-4(11/3) =$	$-46/3$	≤ 0	x_6

```
X1 =  0.67-  0.33 X3 +  0.33 X4
X2 =  3.67+  0.67 X3 +  0.33 X4
X5 =  6.33+  1.33 X3 +  0.67 X4
X6 = 15.33+  2.33 X3 +  1.67 X4
--------------------------------
z  = -9.33-  0.33 X3 -  1.67 X4
```

Conclusion: $y_3 = y_4 = 0$.

What can we conclude by looking at the x_i's that are non-zero, i = 1, 2, … , n?

x_i= -1* (coefficient of the ith slack y_{m+i} for the dual problem's last dictionary).

So if x_i is non-zero, y_{m+i} cannot be in the basis in the last dual dictionary since otherwise it has an implicit 0 coefficient. So y_{m+i} = 0 and equation i of the dual hence must be tight.

Conclusion: if x_i is non-zero, equation i of the dual must be tight.

```
The initial dictionary:
Y5= 3 + 1 Y1 -2 Y2 + 0 Y3 -1 Y4
Y6= 2 - 1 Y1 -1 Y2 - 2 Y3 -4 Y4
----------------------------------
z = 0 + 3 Y1 +5 Y2 + 1 Y3 +0 Y4

After  2 pivots:
Y2= 1.67 -0.67 Y3  -1.67 Y4 -0.33 Y5 -0.33 Y6
Y1= 0.33 -1.33 Y3  -2.33 Y4 +0.33 Y5 -0.67 Y6
---------------------------------------------
z = 9.33 -6.33 Y3 -15.33 Y4 -0.67 Y5 -3.67 Y6

X1=  2/3   [0.67]
X2= 14/3   [3.67]
```

```
X1 =  0.67-  0.33 X3 +  0.33 X4
X2 =  3.67+  0.67 X3 +  0.33 X4
X5 =  6.33+  1.33 X3 +  0.67 X4
X6 = 15.33+  2.33 X3 +  1.67 X4
-------------------------------
z  = -9.33-  0.33 X3 -  1.67 X4
```

Since x_1 and x_2 are non-zero, both dual equations must be tight:

$$-1Y_1 + 2Y_2 + 0Y_3 + 1Y_4 \leq 3$$
$$1Y_1 + 1Y_2 + 2Y_3 + 4Y_4 \leq 2$$

Solving this:

$$-1Y_1 + 2Y_2 + 0Y_3 + 1Y_4 = 3$$
$$1Y_1 + 1Y_2 + 2Y_3 + 4Y_4 = 2$$

$y_3 = y_4 = 0$.

Add them together:

$3\ Y_2 = 5$ so $Y_2 = 5/3$

$Y_1 + Y_2 = 2$ so $Y_1 = 1/3$

The idea for complementary slackness:

1. If $x_{n+i} \neq 0$ for some $i= 1, 2, \dots, m$, then x_{n+i} is in the basis and hence has an implicit 0 coefficient in the z row of the last dictionary and this implies that $y_i=0$.

2. If $x_i \neq 0$ for some $i= 1, 2, \dots, n$, then the ith dual equation is tight ($y_{m+i}=0$) because otherwise, if $y_{m+i} \neq 0$, it is in the basis of the last dual dictionary and hence would have an implicit 0 coefficient in the z row of the last dual dictionary and this would imply $x_i=0$.

In other words:

1. If equation i of the primal is not tight, $y_i=0$.

2. If $x_i \neq 0$, i= 1, 2, … , n the ith dual equation is tight.

Complementary Slackness Theorem (5.2):
x' = feasible primal solution
y' = feasible dual solution
Necessary and sufficient conditions for simultaneous optimality of x' and y' are

1. For all i=1, 2, …, m, either $(a_{i1}* x_1' + a_{i2}*x_2' + \ldots + a_{in}*x_n') = b_i$ or $y_i' = 0$ (or both).

2. For all i=1, 2, ..., n, either $(a_{1j}* y_1' + a_{2j}*y_2' + \ldots + a_{mj}*y_m') = c_i$ or $x_i' = 0$ (or both).

We have already argued that these conditions are necessary for optimality.

Why are they sufficient?