My grad student was doing research for me and printed the output for a problem but unfortunately lost the last page of the output that had the final z row on it.

I want to know the solution to the dual problem. Which variables from the dual problem must be 0? Explain why.

```
X5= 35 - 1.0 X1 -1.0 X3 +0.0 X6 +0.0 X7
X2= 30 - 2.0 X1 +0.0 X3 -1.0 X6 +0.0 X7
X4=  5 + 0.5 X1 -2.5 X3 +0.5 X6 -0.5 X7
X8= 90 - 9.5 X1 -2.5 X3 -3.5 X6 +0.5 X7
```

Summary of complementary slackness:

1. If $x_i \neq 0$, i= 1, 2, … , n the ith dual equation is tight.

2. If equation i of the primal is not tight, $y_i=0$.

The problem from the previous slide was:

Maximize $1\ X1 + 2\ X2 + 1\ X3 + 2\ X4$

subject to

$1\ X1 + 0\ X2 + 1\ X3 + 0\ X4 \le 35$

$2\ X1 + 1\ X2 + 0\ X3 + 0\ X4 \le 30$

$1\ X1 + 1\ X2 + 5\ X3 + 2\ X4 \le 40$

$3\ X1 - 3\ X2 + 5\ X3 + 1\ X4 \le 5$

$X1, X2, X3, X4 \ge 0$

The solution had

$X1 = 0,\ X2 = 30,\ X3 = 0,\ X4 = 5$

Finish solving for the dual solution using complementary slackness.

Checking our algebra with the last dictionary:

```
After   5 pivots:
X5= 35 - 1.0 X1 -1.0 X3 +0.0 X6 +0.0 X7
X2= 30 - 2.0 X1 +0.0 X3 -1.0 X6 +0.0 X7
X4=  5 + 0.5 X1 -2.5 X3 +0.5 X6 -0.5 X7
X8= 90 - 9.5 X1 -2.5 X3 -3.5 X6 +0.5 X7
------------------------------------------------------------------------
z = 70 - 2.0 X1 -4.0 X3 -1.0 X6 -1.0 X7
```

Complementary Slackness Theorem (5.2):

x' = feasible primal solution

y' = feasible dual solution

Necessary and sufficient conditions for simultaneous optimality of x' and y' are

1. For all i=1, 2, ..., m, either $(a_{i1}* x_1' + a_{i2}*x_2' + ... + a_{in}*x_n')= b_i$ or $y_i' = 0$ (or both).

2. For all i=1, 2, ..., n, either $(a_{1j}* y_1' + a_{2j}*y_2' + ... + a_{mj}*y_m') = c_i$ or $x_i' = 0$ (or both).

Recipe for Using this Theorem

Given: x' which is a feasible solution.

Question: Is x' an optimal solution?

Procedure:

1. First consider each x_j' (j=1, 2, ..., n) such that x_j' is NOT 0. Write down the equation of the dual that corresponds to the coefficients of x_j in the primal: $(a_{1j}\, y_1 + a_{2j}\, y_2 + \ldots + a_{mj}\, y_m) = c_j$

2. Next consider each equation i of the primal problem (i=1, 2, 3, ... , m). If the ith equation is NOT tight (there is some slack), write down $y_i = 0$.

3. Solve these equations that you get for y. If there is a unique solution continue. If not, abort.

4. Test y to see if it is dual feasible. Yes- x' is optimal. No- x' is not optimal.

When does the system of equations have a unique solution?

Theorem 5.4 in text: when x' is a nondegenerate basic feasible solution.

Nondegenerate: no basic variables have value 0.

The problem:

Maximize $5\ X_1 + 4\ X_2 + 3\ X_3$

subject to

$$
\begin{aligned}
2\ X_1 + 3\ X_2 + 1\ X_3 &\leq 5 \\
4\ X_1 + 1\ X_2 + 2\ X_3 &\leq 11 \\
3\ X_1 + 4\ X_2 + 2\ X_3 &\leq 8
\end{aligned}
$$

$X_1,\ X_2,\ X_3 \geq 0$

The second last dictionary:
u= (2.5, 0, 0)

```
X1 =2.5 -1.5 X2 -0.5 X3 -0.5  X4
X5 = 1  + 5  X2 + 0  X3 + 2   X4
X6 =0.5 +0.5 X2 -0.5 X3 +1.5  X4
--------------------------------
z= 12.5 -3.5 X2 +0.5 X3 -2.5 X4
```

Check this using complementary slackness.

$u = (2.5, \quad 0, \quad 0)$

Maximize

$5 X_1 + 4 X_2 + 3 X_3$

Constraints:

$2 X_1 + 3 X_2 + 1 X_3 \leq 5$

$4 X_1 + 1 X_2 + 2 X_3 \leq 11$

$3 X_1 + 4 X_2 + 2 X_3 \leq 8$

1. First consider each X_j' such that X_j' is NOT 0. Write down the equation of the dual that corresponds to the coefficients of X_j in the primal:

$X_1 \neq 0$: $2 Y_1 + 4 Y_2 + 3 Y_3 = 5$

2. Next consider each equation i of the primal problem If the ith equation is NOT tight (there is some slack), write down $Y_i = 0$.

$$2\ X_1 + 3\ X_2 + 1\ X_3 \le 5$$
$$4\ X_1 + 1\ X_2 + 2\ X_3 \le 11$$
$$3\ X_1 + 4\ X_2 + 2\ X_3 \le 8$$

$2(2.5) + 3(0) + 1(0) = 5 = 5$ TIGHT

$4(2.5) + 1(0) + 2(0) = 10 < 11$ $Y_2=0$

$3(2.5) + 4(0) + 2(0) = 7.5 < 8$ $Y_3=0$

Solve for y:

$$2\ Y_1 + 4\ Y_2 + 3\ Y_3 = 5$$
$$Y_2 = 0$$
$$Y_3 = 0$$

$y_1 = 5/2$, $y_2=0$, $y_3=0$.

The problem:

Maximize $5\ X_1 + 4\ X_2 + 3\ X_3$
subject to

$$
\begin{aligned}
2\ X_1 + 3\ X_2 + 1\ X_3 &\leq 5 \\
4\ X_1 + 1\ X_2 + 2\ X_3 &\leq 11 \\
3\ X_1 + 4\ X_2 + 2\ X_3 &\leq 8
\end{aligned}
$$

$X_1,\ X_2,\ X_3 \geq 0$

What is the dual?
Check y1= 5/2, y2=0, y3=0 for dual feasibility.

The last dictionary:
v= (2, 0, 1)

```
X1 =   2 - 2 X2 - 2 X4 + 1 X6
X5 =   1 + 5 X2 + 2 X4 + 0 X6
X3 =   1 + 1 X2 + 3 X4 - 2 X6
-------------------------------
z  =  13 - 3 X2 - 1 X4 - 1 X6
```

Check this using complementary slackness.

1. Write down dual equations for non-zero X_i's:

v= (2, 0, 1)

Maximize

$$5 X_1 + 4 X_2 + 3 X_3$$

Constraints:

$$2 X_1 + 3 X_2 + 1 X_3 \leq 5$$
$$4 X_1 + 1 X_2 + 2 X_3 \leq 11$$
$$3 X_1 + 4 X_2 + 2 X_3 \leq 8$$

$X_1 \neq 0$: $2 Y_1 + 4 Y_2 + 3 Y_3 = 5$

$X_3 \neq 0$: $1 Y_1 + 2 Y_2 + 2 Y_3 = 1$

2. Next consider each equation i of the primal problem If the ith equation is NOT tight (there is some slack), write down $Y_i = 0$.

v= (2, 0, 1)

Constraints:

$$2X_1 + 3X_2 + 1X_3 \leq 5$$
$$4X_1 + 1X_2 + 2X_3 \leq 11$$
$$3X_1 + 4X_2 + 2X_3 \leq 8$$

2(2) + 3(0) + 1(1)= 5 = 5 TIGHT

4(2) + 1(0) + 2(1)= 10< 11 Y_2=0

3(2) + 4(0) + 2(1)= 8= 8 TIGHT

3. Solve for Y.

$$X_1 \neq 0: \quad 2\ Y_1 + 4\ Y_2 + 3\ Y_3 = 5$$
$$X_3 \neq 0: \quad 1\ Y_1 + 2\ Y_2 + 2\ Y_3 = 3$$
$$Y_2 = 0$$

Solution: (1, 0, 1)

Test y for dual feasibility.

We have already argued that these conditions are necessary for optimality.

Why are they sufficient?

Complementary Slackness Theorem (5.2):

x' = feasible primal solution

y' = feasible dual solution

Necessary and sufficient conditions for simultaneous optimality of x' and y' are

1. For all i=1, 2, ..., m, either $(a_{i1}* x_1' + a_{i2}*x_2' + ... + a_{in}*x_n')= b_i$ or $y_i' = 0$ (or both).

2. For all i=1, 2, ..., n, either $(a_{1j}* y_1' + a_{2j}*y_2' + ... + a_{mj}*y_m') = c_i$ or $x_i' = 0$ (or both).

Earlier we showed that for feasible primal and dual solutions:

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \le S =$$

$$(a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m) x_1 +$$
$$(a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m) x_2 +$$
$$\ldots +$$
$$(a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m) x_n$$

$$=$$

$$(a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n) y_1 +$$
$$(a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n) y_2 +$$
$$\ldots +$$
$$(a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n) y_m$$

$$\le b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

In a final dictionary:
m basic variables.
Of these, assume b of them are from $x_1, \ldots, x_n$ and the remaining m-b are slacks.
If the solution is not degenerate then we get b equations from having the non-zero values from $x_1, \ldots, x_n$ and m-b equations that are not tight.

This gives a square (b by b) system to solve.

Complementary Slackness Theorem (5.2):
x' = feasible primal solution
y' = feasible dual solution
Necessary and sufficient conditions for simultaneous optimality of x' and y' are

1. For all i=1, 2, ..., m, either $(a_{i1}* x_1' + a_{i2}*x_2' + ... + a_{in}*x_n')= b_i$ or $y_i' = 0$ (or both).

2. For all i=1, 2, ..., n, either $(a_{1j}* y_1' + a_{2j}*y_2' + ... + a_{mj}*y_m') = c_i$ or $x_i' = 0$ (or both).

Problem is the “or both” parts.