Use complementary slackness to check the solution:

(20/3, 0 , 16/3, 0)

Maximize

$$9 x_1 - 3 x_2 + 3 x_3 - 7 x_4$$

subject to

$$\begin{aligned} 2 x_1 + 8 x_2 - 1 x_3 + 11 x_4 &\le 8 \\ 1 x_1 + 1 x_2 + 1 x_3 + 1 x_4 &\le 13 \\ 1 x_1 + 4 x_2 + 1 x_3 + 3 x_4 &\le 12 \\ -1 x_1 + 2 x_2 + 1 x_3 + 3 x_4 &\le -1 \end{aligned}$$

$$x_1, x_2, x_3, x_4 \ge 0$$

Recipe for Using this Theorem

Given: x' which is a feasible solution.

Question: Is x' an optimal solution?

Procedure:

1. First consider each x_j' (j=1, 2, ..., n) such that x_j' is NOT 0. Write down the equation of the dual that corresponds to the coefficients of x_j in the primal: $(a_{1j}\, y_1 + a_{2j}\, y_2 + \ldots + a_{mj}\, y_m) = c_j$

2. Next consider each equation i of the primal problem (i=1, 2, 3, ... , m). If the ith equation is NOT tight (there is some slack), write down $y_i = 0$.

3. Solve these equations that you get for y. If there is a unique solution continue. If not, abort.

4. Test y to see if it is dual feasible. Yes- x' is optimal. No- x' is not optimal.

When does the system of equations have a unique solution?

Theorem 5.4 in text: when x' is a nondegenerate basic feasible solution.

Nondegenerate: no basic variables have value 0.

To check: read dual solution from final dictionary:

```
X8 =  0.33-  6.00 X2 -  9.33 X4 -  0.67 X5 +  0.33 X7
X6 =  1.00+  3.00 X2 +  2.00 X4 +  0.00 X5 +  1.00 X7
X3 =  5.33+  0.00 X2 +  1.67 X4 +  0.33 X5 -  0.67 X7
X1 =  6.67-  4.00 X2 -  4.67 X4 -  0.33 X5 -  0.33 X7
----------------------------------------------
z  = 76.00- 39.00 X2 - 44.00 X4 -  2.00 X5 -  5.00 X7
```

How do we check these two solutions using duality theory?

I plan to take your midterm score out of 90 instead of 100.

Programming Project 2.
Due at 11:55pm on Fri. Oct. 24.
Late submissions accepted with 10% penalty until Tues. Oct. 28 at 11:55pm.

CSC 545 only:
Survey Paper- due on Fri. Oct. 17 at 11:55pm.
Late submissions accepted until Tues. Oct. 21 at 11:55pm with a 10% late penalty.

Suppose you are solving a LP problem for an industrial partner, have found an optimal solution to their problem, but then a change to the b_i's is desired. Is is necessary to resolve the problem? The answer is NO if the change to the b_i's is sufficiently small (defined later).

Forestry Example:

A forester has 100 acres of land and $4000 in capital and would like to plant X_1 acres of hardwood and X_2 acres of pine to maximize profit where per acre:

Wood type	Cost to harvest	Selling price	Profit
Hardwood	10	50	40
Pine	50	120	70

What is the linear programming problem?

The linear programming problem is:

Maximize $40 X_1 + 70 X_2$

subject to

$$X_1 + X_2 \leq 100$$
$$10 X_1 + 50 X_2 \leq 4000$$

$$X_1, X_2 \geq 0$$

The initial dictionary is:

```
X3 =   100 - 1 X1 - 1 X2
X4 =  4000 -10 X1 -50 X2
--------------------------
z  =     0 +40 X1 +70 X2
```

The final dictionary is:

```
X1 = 25   - 1.25 X3 + 0.025 X4
X2 = 75   + 0.25 X3 - 0.025 X4
-------------------------------
z  = 6250 -32.50 X3 - 0.750 X4
```

Dual solution: $Y_1 = 32.5$, $Y_2 = 0.75$

The initial dictionary is:

```
X3 =   100 - 1 X1 - 1 X2   * Y1
X4 =  4000 -10 X1 -50 X2   * Y2
--------------------------
z  =     0 +40 X1 +70 X2
```

From our results on duality, we know that the z row is equal to:

```
                 0 + 40X1 + 70 X2
+ (32.5) *[ 100 - 1 X1 -  1 X2 - X3]
+ (0.75) *[4000 -10 X1 - 50 X2 - X4]
---------------------------------
=    6250.00  -32.50 X3 - 0.75 X4
```

Suppose that t_1 extra acres and t_2 extra dollars are provided. The problem then becomes:

Maximize $40 X1 + 70 X2$

subject to

$$X1 + X2 \leq 100 + t_1$$
$$10 X1 + 50 X2 \leq 4000 + t_2$$

$X1, X2 \geq 0$

If X_1 and X_2 are still the basis elements at the end, then z will be:

$$
\begin{array}{l}
\qquad\qquad\qquad\quad 0 \quad +40\ X1\ +70\ X2 \\
+\ (32.5)*[\ 100+t_1 - 1\ X1\ -\ 1\ X2\ -\ X3] \\
+\ (0.75)*[4000+t_2 -10\ X1\ -50\ X2\ -\ X4] \\
\text{-------------------------------} \\
=\quad 6250\ +\ 32.50\ t_1\ +\ 0.75\ t_2 \\
\qquad -32.50\ X3\ -\ 0.75\ X4
\end{array}
$$

Conclusion (proof later):
If b changes to b + t where t is sufficiently small, then z changes by a factor of y^T t.

How does X change when b changes to b + t?

Consider the equations that we started with:

$$X_1 + X_2 + X_3 = 100$$

$$10\,X_1 + 50\,X_2 + X_4 = 4000$$

In matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 10 & 50 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 100 \\ 4000 \end{bmatrix}$$

Equation A

At the end of the computation:

```
X1 =  25   - 1.25 X3 + 0.025 X4
X2 =  75   + 0.25 X3 - 0.025 X4
------------------------------
z  = 6250 -32.50 X3 - 0.75 X4
```

In matrix form (Equation B):

$$\begin{bmatrix} 1 & 0 & 1.25 & -0.025 \\ 0 & 1 & -0.25 & 0.025 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 25 \\ 75 \end{bmatrix}$$

At the end X_1, X_2 is the basis. Take the columns that correspond to the basis in the original matrix and make a square matrix B with basis headers:

$$B = \begin{bmatrix} 1 & 1 \\ 10 & 50 \end{bmatrix} \quad (\text{columns } X_1, X_2) \qquad B^{-1} = \begin{bmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{bmatrix}$$

To get from Eq. A to Eq. B, multiply both the LHS, and the RHS by B^{-1}.

$$B^{-1}\begin{bmatrix}1 & 1 & 1 & 0\\ 10 & 50 & 0 & 1\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\\ X_4\end{bmatrix} = B^{-1}\begin{bmatrix}100\\ 4000\end{bmatrix}$$

If b changes:

$$B^{-1}\begin{bmatrix}1 & 1 & 1 & 0\\ 10 & 50 & 0 & 1\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\\ X_4\end{bmatrix} = B^{-1}\begin{bmatrix}100 + t1\\ 4000 + t2\end{bmatrix}$$

If b changes:

$$B^{-1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 10 & 50 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = B^{-1} \begin{bmatrix} 100 + t1 \\ 4000 + t2 \end{bmatrix}$$

and the change is not so big that the basis changes in the optimal solution, the new value for X is:

B-1 * (x + t) =

[X1] = [5/4 -1/40] [100 + t1]
[X2] [-1/4 1/40] [4000 + t2]

X1 = 25 +5 t1 /4 -t2/40

X2 = 75 -1 t1 /4 +t2/40

$$B^{-1}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 10 & 50 & 0 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = B^{-1}\begin{bmatrix} 100 + t1 \\ 4000 + t2 \end{bmatrix}$$

Solution to this is X.

If the change is not so big that the basis changes in the optimal solution, the new value for X' is:

$X' = X + B^{-1}\ t =$

$$\begin{bmatrix} X1 \\ X2 \end{bmatrix} = \begin{bmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{bmatrix} \begin{bmatrix} 100 + t_1 \\ 4000 + t_2 \end{bmatrix}$$

$X1 = 25 + 5/4\ t_1 \quad -1/40\ t_2$

$X2 = 75 - 1/4\ t_1 \quad +1/40\ t_2$

This equation only gives the correct solution when X1, X2 is the correct basis at the end.

The final dictionary would be:

$$X1 = 25 + (5/4)\, t_1 - (1/40) t_2 - 1.25\, X3 + 0.025\, X4$$

$$X2 = 75 - (1/4)\, t_1 + (1/40)\, t_2 + 0.25\, X3 - 0.025\, X4$$

$$z = 6250 + 32.5\, t_1 + 0.75\, t_2 - 32.50\, X3 - 0.75\, X4$$

When is this feasible?

When is this feasible?

$X_1 = 25 + (5/4)\ t_1\ - (1/40)\ t_2\ - \ldots$

$X_2 = 75\ - (1/4)\ t_1\ + (1/40)\ t_2\ - \ldots$

This is not a correct choice for the basis if $X_1 < 0$ or $X_2 < 0$.

If $t_1=0$, and $[\ 25 - (1/40)t_2] < 0$ (or equivalently, $t_2 > 1000$), X_1 moves out of the basis.

If $t_2=0$, and $[75 - (1/4)t_1] < 0$ (or equivalently $t_1 > 300$), then X_2 moves out of the basis.

Forestry Example: 100 acres, $4000

Wood type	Cost to harvest	Selling price	Profit	Solution
Hardwood	10	50	40	25
Pine	50	120	70	75

$t_1 > 300$: $X_1 + X_2 \leq 100 + t_1$

If the forester had 400 acres (300 more) then he could afford to grow and harvest all hardwood.

If he had > 300 more acres, the extra land is wasted (no money to harvest). The slack for the acres equation enters the basis, the variable for the number of acres of pine leaves.

Forestry Example: 100 acres, \$4000

Wood type	Cost to harvest	Selling price	Profit	Solution
Hardwood	10	50	40	25
Pine	50	120	70	75

$t_2 > 1000$: $10\ X_1 + 50\ 2 \leq 4000 + t_2$

Pine provides more profit. With \$5000 (\$1000 more) the forester could grow all pine.

If the forester has > \$5000, the extra money is not useful. The slack variable for the money equation enters the basis and the variable for number of acres of hardwood leaves.