Question from last class:

If you have a primal feasible solution x, and solve for y using duality theory, does having $c^T x = b^T y$ guarantee optimality for x?

Check to see if this holds for

x= (3, 0)

Maximize

$$2 x_1 + 9 x_2$$

subject to

$$1 x_1 - 1 x_2 \leq 3$$

$$1 x_1 + 1 x_2 \leq 9$$

$x_1, x_2 \geq 0$

“The challenge of maximizing satisfaction within a set of limits makes linear programming an ideal tool of analysis for economics, which studies the ways in which households, businesses and societies allocate limited resources to achieve needs and wants.”

Shane Hall: http://www.ehow.com/about_6329269_linear-programming-economic-analysis.html

Manufacturing Problems

Suppose that:

- The problem is to maximize profit.
- Each x_j measures the level of output of the jth product.
- Each b_i specifies available amount of ith resource.
- Each a_{ij} is the units of resource i required for product j.
- Each c_j is profit in dollars per unit of product j.

How should the dual variables be interpreted in order for the equations to make sense?

Forestry Example: 100 acres, $4000

Wood type	Cost to harvest	Selling price	Profit	Solution
Hardwood	10	50	40	25
Pine	50	120	70	75

Maximize Profit

$$40\ X_1 + 70\ X_2$$

subject to

$$X_1 + X_2 \le 100$$

$$10\ X_1 + 50\ X_2 \le 4000$$

$$X_1,\ X_2 \ge 0$$

Product 1 : Hardwood.
Product 2 : Pine.
Resource 1 : Acres of land.
Resource 2 : Dollars.

The primal:

Maximize $c^T x$
subject to
$Ax \le b$,
$x \ge 0$.

The dual:

Minimize $b^T y$
subject to
$A^T y \ge c$
$y \ge 0$.

- Each a_{ij} is the units of resource i required for product j.
- Each c_j is profit in dollars per unit of product j.

$$a_{1i} \; y_1 + a_{2i} y_2 + \dots + a_{mi} \; y_m \geq c_i$$

[Units res. 1 for product i][? y_1] +
[Units res. 2 for product i][? y_2] +
...
[Units res. m for product i][? y_m]

≥ [Dollars per unit of product i]

•Each a_{ij} is the units of resource i required for product j.

•Each c_j is profit in dollars per unit of product j.

$$a_{1i} \; y_1 + a_{2i} y_2 + \ldots + a_{mi} \; y_m \geq c_i$$

To make this work, we need:

[Units res. 1/prod. i][Dollars/Unit res.1]+
[Units res. 2/prod. i][Dollars/Unit res.2]+
...
[Units res. m/prod. i][Dollars/Unit res. m]
≥
[Dollars / prod. i]

So, y_i is expressed in terms of dollars per unit of resource i.

Let y^* be the optimal solution to the dual.
With each extra unit of resource i, the profit increases by y_i^* dollars.
So, y_i^* represents the extra amount the firm should be willing to pay over and above the current trading price for each extra unit of resource i.

Definitions

$y=^*$ is called the *marginal value* of the ith resource.
Marginal- difference between trading price and actual worth.
y_i^* is also called the *shadow price* of the ith resource.

Maximize Profit

$40\ X_1 + 70\ X_2$

subject to

$X_1 + X_2 \leq 100$

$10\ X_1 + 50\ X_2 \leq 4000$

Product 1 : Hardwood.

Product 2 : Pine.

Resource 1 : Acres of land.

Resource 2 : Dollars.

Looking at the Units:

hard= acres of hardwood, pine= acres of pine.

Maximize

(profit/hard)(hard) + (profit/pine) (pine) =profit

subject to

(acres/hard) (hard) + (acres/pine) (pine) $\leq$ acres

(dollars/hard)(hard) + (dollars/pine)(pine) $\leq$ dollars

Looking at the DUAL Units:

hard= acres of hardwood,

pine= acres of pine.

Maximize

(profit/hard) (hard)+ (profit/pine) (pine) =profit

subject to

(acres/hard) (hard) + (acres/pine) (pine)

$\leq$ acres

(dollars/hard) (hard) + (dollars/pine) (pine)

$\leq$ dollars

Looking at the DUAL Units:

hard= acres of hardwood,

pine= acres of pine.

The units are:

y_1 : profit/acre

y_2 : profit/dollar

Maximize

(profit/hard) (hard)+ (profit/pine) (pine) =profit

subject to

(acres/hard) (hard) + (acres/pine) (pine) $\leq$ acres

(dollars/hard) (hard) + (dollars/pine) (pine) $\leq$ dollars

Dual solution: $y_1 = 32.5$, $y_2 = 0.75$

y_1 : profit/acre, y_2 : profit/dollar

Conclusion:

One extra acre would result in $32.50 extra profit.

One extra dollar in capital results in 0.75 profit.

Renting one acre at less than $32.50 results in more profit.

Borrowing money at less than 0.75 interest rate results in more profit.

We already covered the update formulas for this problem when there is a change to the bi's.