```
Initial dictionary:
X4 =5  -1 X1 -1 X2 +1 X3
X5 =3  -1 X1 +0 X2 -2 X3
X6 =4  +0 X1 -1 X2 -6 X3
------------------------
z = 0  +1 X1 +3 X2 +6 X3
```

What are B and B^{-1} for the final dictionary?

```
Final dictionary:
X2 =4 -6 X3 +0 X4 -1 X6
X1 =1 +7 X3 -1 X4 +1 X6
X5 =2 -9 X3 +1 X4 -1 X6
-----------------------
z =13 -5 X3 -1 X4 -2 X6
```

The initial matrix A:

	X1	X2	X3	X4	X5	X6	
[	1	1	-1	1	0	0	]
[	1	0	2	0	1	0	]
[	0	1	6	0	0	1	]

The final matrix:

	X1	X2	X3	X4	X5	X6	
[	0	1	6	0	0	1	]
[	1	0	-7	1	0	-1	]
[	0	0	9	-1	1	1	]

The initial matrix A:

	X1	X2	X3	X4	X5	X6	
[	1	1	-1	1	0	0	]
[	1	0	2	0	1	0	]
[	0	1	6	0	0	1	]

The final matrix:

	X1	X2	X3	X4	X5	X6	
[	0	1	6	0	0	1	]
[	1	0	-7	1	0	-1	]
[	0	0	9	-1	1	1	]

B =

	X2	X1	X5	
[	1	1	0	]
[	0	1	1	]
[	1	0	0	]

B^{-1} =

	X4	X5	X6	
[	0	0	1	]
[	1	0	-1	]
[	-1	1	1	]

Initial dictionary:

```
X4 =5  -1 X1 -1 X2 +1 X3
X5 =3  -1 X1 +0 X2 -2 X3
X6 =4  +0 X1 -1 X2 -6 X3
------------------------
z = 0  +1 X1 +3 X2 +6 X3
```

What of this information do we really need to know to decide where to pivot?

Initial dictionary:

```
X4 =5  -1 X1 -1 X2 +1 X3
X5 =3  -1 X1 +0 X2 -2 X3
X6 =4  +0 X1 -1 X2 -6 X3
------------------------
z = 0  +1 X1 +3 X2 +6 X3
```

1. Look at z row to choose the pivot column.

With smallest subscript rule, we would choose X1 to enter.

```
Initial dictionary:
X4 =5  -1 X1 -1 X2 +1 X3
X5 =3  -1 X1 +0 X2 -2 X3
X6 =4  +0 X1 -1 X2 -6 X3
------------------------
z = 0  +1 X1 +3 X2 +6 X3
```

2. Look at entering column and b column to find tightest constraint (pivot row).

```
After   1 pivot:

X4 =  2-  1 X2 +  3 X3 +  1 X5
X1 =  3+  0 X2 -  2 X3 -  1 X5
X6 =  4-  1 X2 -  6 X3 +  0 X5
-------------------------------
z  =  3+  3 X2 +  4 X3 -  1 X5
```

Idea behind the revised Simplex method:

just keep track of what we need to know instead of the entire tableau.

Note: A standard convention in matrix algebra is that a vector x is by default a column vector and x^T is a row vector. The text does not follow this standard convention, but I am using it. Hence, the notation here may differ a bit from the text.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{bmatrix} \qquad x^T = [x_1\ x_2\ x_3\ \ldots\ x_n]$$

In the Simplex Method, what information is needed to find the next basic feasible solution?

1. Coefficients of non-basic variables in the z row to determine the pivot column.

2. The pivot column and the current solution to determine the pivot row. The new solution is then determined from the pivot column and the old solution.

H_B^T = indices of basic columns.
H_N^T = indices of non-basic columns.
Split the matrix A into two parts:
A_B = columns corresponding to H_B^T.
A_N = columns corresponding to H_N^T.
Create c_B, c_N, and x_B, x_N the same way.

What are these for this initial dictionary?

```
X4 =5  -1 X1 -1 X2 +1 X3
X5 =3  -1 X1 +0 X2 -2 X3
X6 =4  +0 X1 -1 X2 -6 X3
------------------------
z = 0  +1 X1 +3 X2 +6 X3
```

H_B^T = indices of basic columns
H_N^T = indices of non-basic columns
Split the matrix A into two parts:
A_B = columns corresponding to H_B^T
A_N = columns corresponding to H_N^T
Create c_B, c_N, and x_B, x_N the same way.
What are these for this final dictionary?

```
X2 =4 -6 X3 +0 X4 -1 X6
X1 =1 +7 X3 -1 X4 +1 X6
X5 =2 -9 X3 +1 X4 -1 X6
-----------------------
z =13 -5 X3 -1 X4 -2 X6
```

Another problem:

```
X4 =   5 - 2 X1 - 3 X2 - 1 X3
X5 =  11 - 4 X1 - 1 X2 - 2 X3
X6 =   8 - 3 X1 - 4 X2 - 2 X3
------------------------------
z  =   0 + 5 X1 + 4 X2 + 3 X3
```

X1 enters. X4 leaves.

```
X1 = 2.5 - 1.5 X2 -0.5 X3 - 0.5 X4
X5 = 1.0 + 5.0 X  +0.0 X3 + 2.0 X4
X6 = 0.5 + 0.5 X2 -0.5 X3 + 1.5 X4
------------------------------------
z = 12.5 - 3.5 X2+ 0.5 X3 - 2.5 X4
```

For this problem we have:

Maximize $c^T x$

$c^T = [5\ 4\ 3\ 0\ 0\ 0]$ $x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^T$

subject to $A x = b$

$$A = \begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}$$

Initially, the basis corresponds to the slack variables x_4, x_5, x_6:

$H_B^T = [\ 4\ 5\ 6]$ // subscripts for basis

The non-basic variables are x_1, x_2, and x_3: $H_N^T = [1\ 2\ 3]$

// subscripts of non-basic variables.

Split the matrix A into two parts:

A_B = columns corresponding to H_B^T

A_N = columns corresponding to H_N^T

H_B^T =

[4 5 6]

A_B =

[1 0 0]

[0 1 0]

[0 0 1]

H_N^T =

[1 2 3]

A_N =

[2 3 1]

[4 1 2]

[3 4 2]

Create c_B, c_N, and x_B, x_N the same way:

$c_B = [0 \ \ 0 \ \ 0]^T$

$c_N = [5 \ \ 4 \ \ 3]^T$

$x_B = [x_4 \ \ x_5 \ \ x_6]^T$

$x_N = [x_1 \ \ x_2 \ \ x_3]^T$

The system of equations can then be expressed as:

(a) $A\ x = b \Longrightarrow A_B\ x_B + A_N\ x_N = b$

(b) $z = c_B^T\ x_B + c_N^T\ x_N$

From (a), $x_B = A_B^{-1}\ b - A_B^{-1}\ A_N\ x_N$

$= A_B^{-1}\ [b - A_N\ x_N]$

Plugging this into the formula for z:

$z = c_B^T\ A_B^{-1}\ [\ b - A_N\ x_N\ ] + c_N^T\ x_N$

From the previous slide:

$$x_B = A_B^{-1} [b - A_N x_N]$$

$$z = c_B^T A_B^{-1} [b - A_N x_N] + c_N^T x_N$$

So the dictionary with basis A_B reads:

$$x_B = A_B^{-1} [b - A_N x_N]$$

$$z = c_B^T A_B^{-1} [b - A_N x_N] + c_N^T x_N$$

The dictionary with basis A_B reads:

$x_B = A_B^{-1} [b - A_N \; x_N]$

$z = c_B^T A_B^{-1} [b - A_N x_N] + c_N^T x_N$

$$x_B = \begin{bmatrix} x4 \\ x5 \\ x6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

```
x4 =  5 - 2 x1 - 3 x2 - 1 x3
x5 = 11 - 4 x1 - 1 x2 - 2 x3
x6 =  8 - 3 x1 - 4 x2 - 2 x3
------------------------------
z  =  0 + 5 x1 + 4 x2 + 3 x3
```

$$z = c_B^T A_B^{-1} [b - A_N x_N] + c_N^T x_N$$

z=

$$(0\ \ 0\ \ 0)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}^{-1}\left(\begin{bmatrix}5\\11\\8\end{bmatrix}-\begin{bmatrix}2 & 3 & 1\\4 & 1 & 2\\3 & 4 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)+$$

$$(5\ \ \ 4\ \ \ 3\ )\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

```
x4 =  5 - 2 x1 - 3 x2 - 1 x3
x5 = 11 - 4 x1 - 1 x2 - 2 x3
x6 =  8 - 3 x1 - 4 x2 - 2 x3
-----------------------------
z  =  0 + 5 x1 + 4 x2 + 3 x3
```

Step 1: Find coefficients of non-basic variables in the z row.

$$z = c_B^T A_B^{-1} [b - A_N x_N] + c_N^T x_N$$

Let $y^T = c_B^T * A_B^{-1}$

Find y^T by solving $y^T A_B = c_B^T$
or equivalently $A_B^T y = c_B$.

Set $z = y^T b + [c_N^T - y^T A_N] x_N$

Find y^T by solving $A_B^T\ y = c_B$.
Set $z = y^T\ b + [c_N^T - y^T\ A_N]\ x_N$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$y_1 = 0,\ y_2 = 0,\ y_3 = 0.$

$$z = [0\ 0\ 0]\begin{bmatrix}5\\11\\8\end{bmatrix} + \left([5\ 4\ 3\ ] - [\ 0\ 0\ 0]\begin{bmatrix}2 & 3 & 1\\4 & 1 & 2\\3 & 4 & 2\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

z = 0 + 5 x1 + 4 x2 + 3 x3

The y^T b gives the constant term in the z row and the rest gives the terms corresponding to the non-basic variables.

Step 2: Determine the leaving variable.

Solve for entering column d in current dictionary: $d = A_B^{-1} a$ where a is the entering column taken from the initial problem.

Or equivalently, solve for d:

$A_B \, d = a$

If tightest constraint corresponds to

$x_{leaving} = v - t * x_{entering}$

then the new value of $x_{entering}$ will be $s = v/t$.

Solving A_B d = a:

Recall: z = 0 + 5 x1 + 4 x2 + 3 x3

Choose x_1 to enter. Which variable leaves?

Solving A_B d= a:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \text{ and so, } \mathrm{d} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

Which equation imposes the tightest constraint?

$$5 - 2x_1 \geq 0 \implies x_1 \leq 5/2 \ (*)$$
$$11 - 4x_1 \geq 0 \implies x_1 \leq 11/4$$
$$8 - 3x_1 \geq 0 \implies x_1 \leq 8/3$$

The first basis variable x_4 leaves because the first equation is the tightest. The value of the entering variable is 5/2 because this is the tightest constraint. Plug in the value of the entering variable:

$$x_4 = 5 - 2(5/2) = 0 \quad \rightarrow \quad x_1 = 5/2$$
$$x_5 = 11 - 4(5/2) = 1$$
$$x_6 = 8 - 3(5/2) = 1/2$$

Updating all the variables:

H_B^T =
[1 5 6]

A_B =
[2 0 0]
[4 1 0]
[3 0 1]

H_N^T =
[4 2 3]

A_N =
[1 3 1]
[0 1 2]
[0 4 2]

c_B = [5 0 0]^T
c_N= [0 4 3]^T
x_B= [x_1 x_5 x_6]^T
x_N= [x_4 x_2 x_3]^T

Current solution:
[5/2 1 1/2]

Z is:
(previous value of Z)
+ (coeff. of entering var. in Z row)
* (new value of entering variable)
= 0 + 5*(5/2) = 25/2

The Revised Simplex Method uses more work to determine the

- z row coefficients,
- the column that should enter,
- and the pivot row number (exiting variable)

but then it takes less work to pivot.

Summary of the steps of the Revised Simplex Method:

Maintain for each step:

1. H_B- the current basis header.
2. H_N- the header for the non-basic variables (optional).
3. the current solution.
4. A_B- the columns from A which correspond to the basis in the same order as in H_B.

[Actually, most programs maintain some factorization of this matrix].

5. A_N- the columns from A which correspond to the basis in the same order as in H_N. [Actually, you can get these columns from A when you need them so you don't really have to store this.]

6. c_B- the costs of the basic variables in the same order as H_B.

7. c_N- the costs of the non-basic variables in the same order as H_N.

8. z- the current value of the objective function.

The Revised Simplex Algorithm
Step 1: Determine pivot column.

Solve $A_B^T y = c_B$ for y.

compute $[c_N^T - y^T A_N] * x_N$
to get coefficients of non-basic variables.

Look for a positive coefficient, say r corresponding to non-basic x_j.

Step 2: Determine the leaving variable.

Solve for entering column d in current dictionary:

$d = A_B^{-1} a$

where a is the entering column taken from the initial problem.

Or equivalently, solve for d:

$A_B \, d = a$

If tightest constraint corresponds to

$x_{leaving} = v - t * x_{entering}$

then the new value of $x_{entering}$ will be $s = v/t$.

Step 3: Update variables (x_j enters, x_k leaves).
Update basic variables headers H_B by replacing k with j. Update H_N by replacing j with k.

Set $x_j = s$ in the new solution. Plug this value for x_j into the other equations to update the values of the other basic variables. The leaving variable will be 0.

Recall r = the coefficient of x_j in the z row:
Set $z = z + r\ s$.

Update A_B, A_N, c_B, c_N, x_B, x_N to match basis headers.

Jacobi's formula:

$A^{-1} = \frac{1}{\det(A)}$ Adj(A) where

$\text{Adj}(A)_{i,j} = (-1)^{i+j} \det(A[j,i])$

and A[j,i] is A with row j and column i deleted.

Use this to find the inverse of:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{bmatrix}$$

Computing the Adjoint matrix:

$$\begin{bmatrix} +\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -4 & 0 & -1 \end{vmatrix} \end{bmatrix}$$