How many faces do you get from walking the faces of this rotation system of K_4?

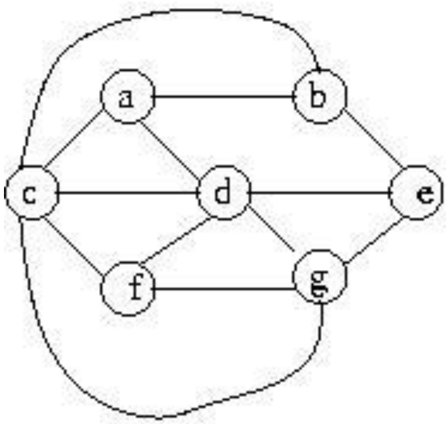
Is this an embedding of K_4 in the plane?

0 : 1 3 2
1 : 0 3 2
2 : 0 3 1
3 : 0 2 1

Rotation Systems

G connected on an orientable surface:

$g = (2 - n + m - f)/2$



a: b d c
b: a c e
c: a d f g b
d: a e g f c
e: b g d
f: c d g
g: c f d e

0 plane

1 torus

2

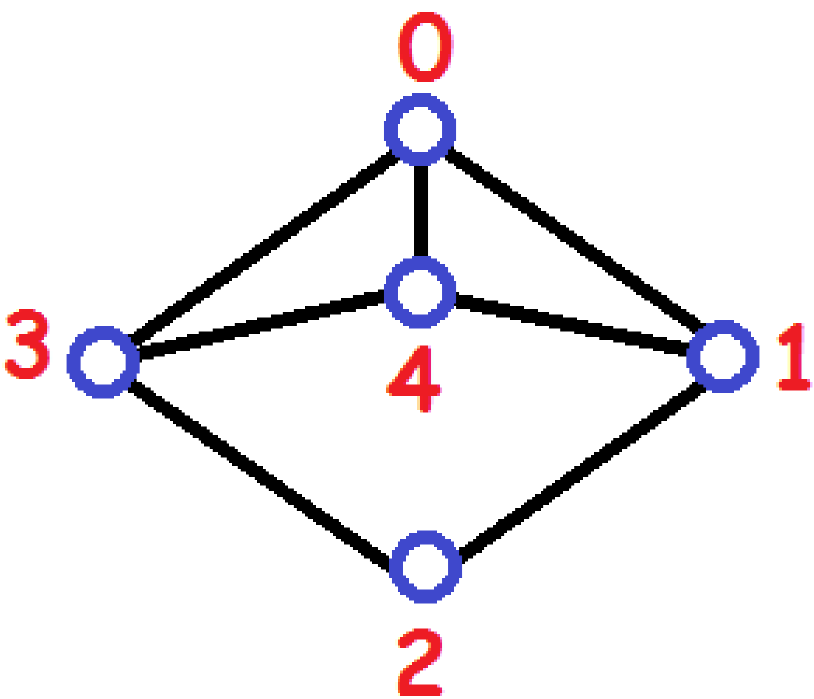
Greg McShane

F0: (a, b)(b, c)(c, a)(a, b)

F1: (a, d)(d, e)(e, b)(b, a)(a, d)

How can we find a rotation system that represents a planar embedding of a graph?

Input graph:
0: 1 3 4
1: 0 2 4
2: 1 3
3: 0 2 4
4: 0 1 3



Planar embedding
0: 1 4 3
1: 0 2 4
2: 1 3
3: 0 4 2
4: 0 1 3

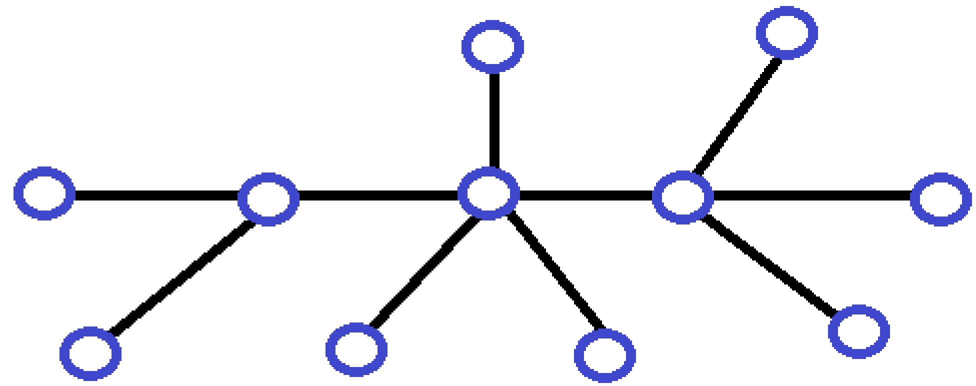
f= number of faces
n= number of vertices
m= number of edges

Euler's formula: For any connected planar graph G, $f = m - n + 2$.

Proof by induction:

How many edges must a connected graph on n vertices have?

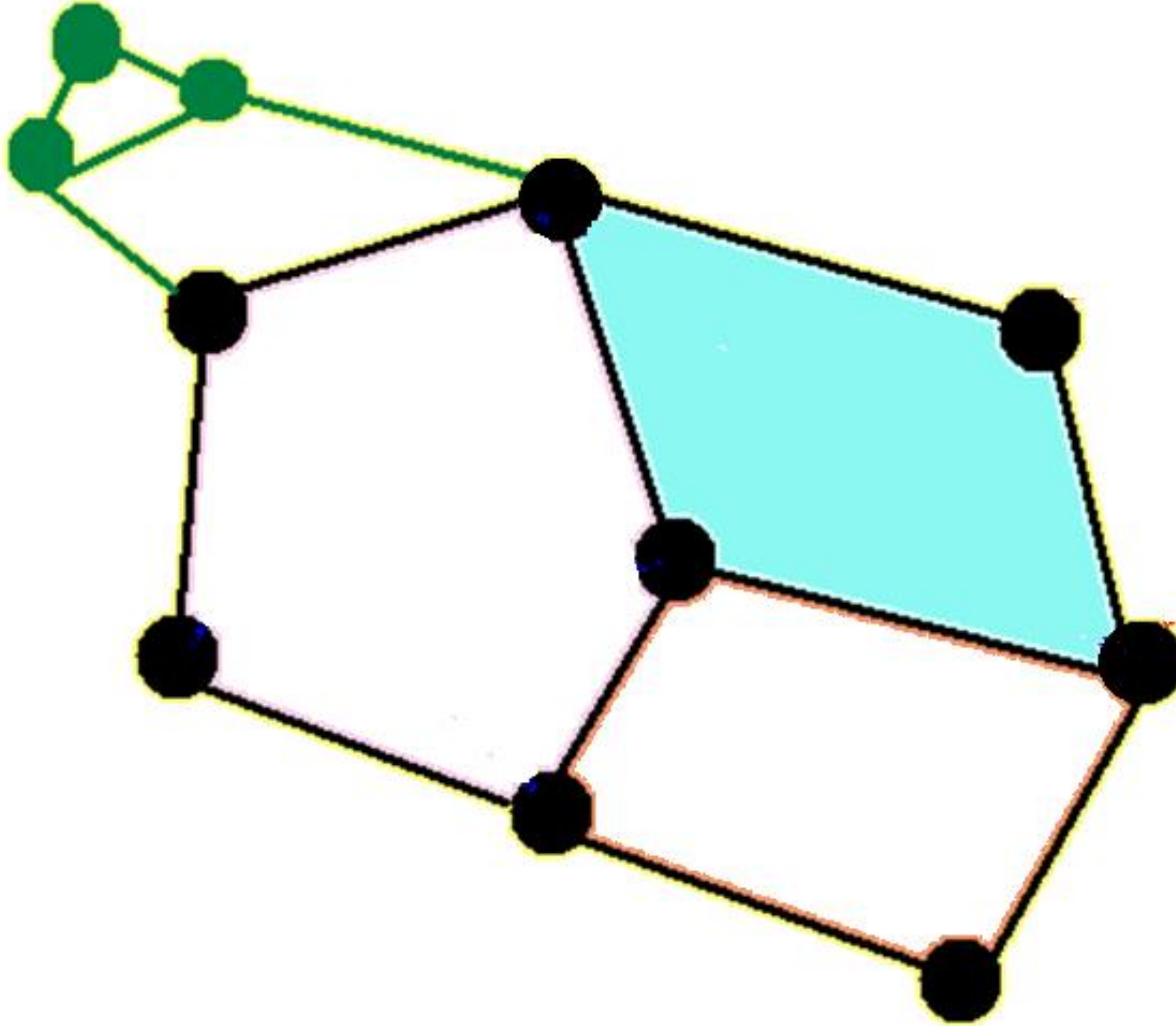
Euler's formula: For any connected planar graph G, f= m-n+2.

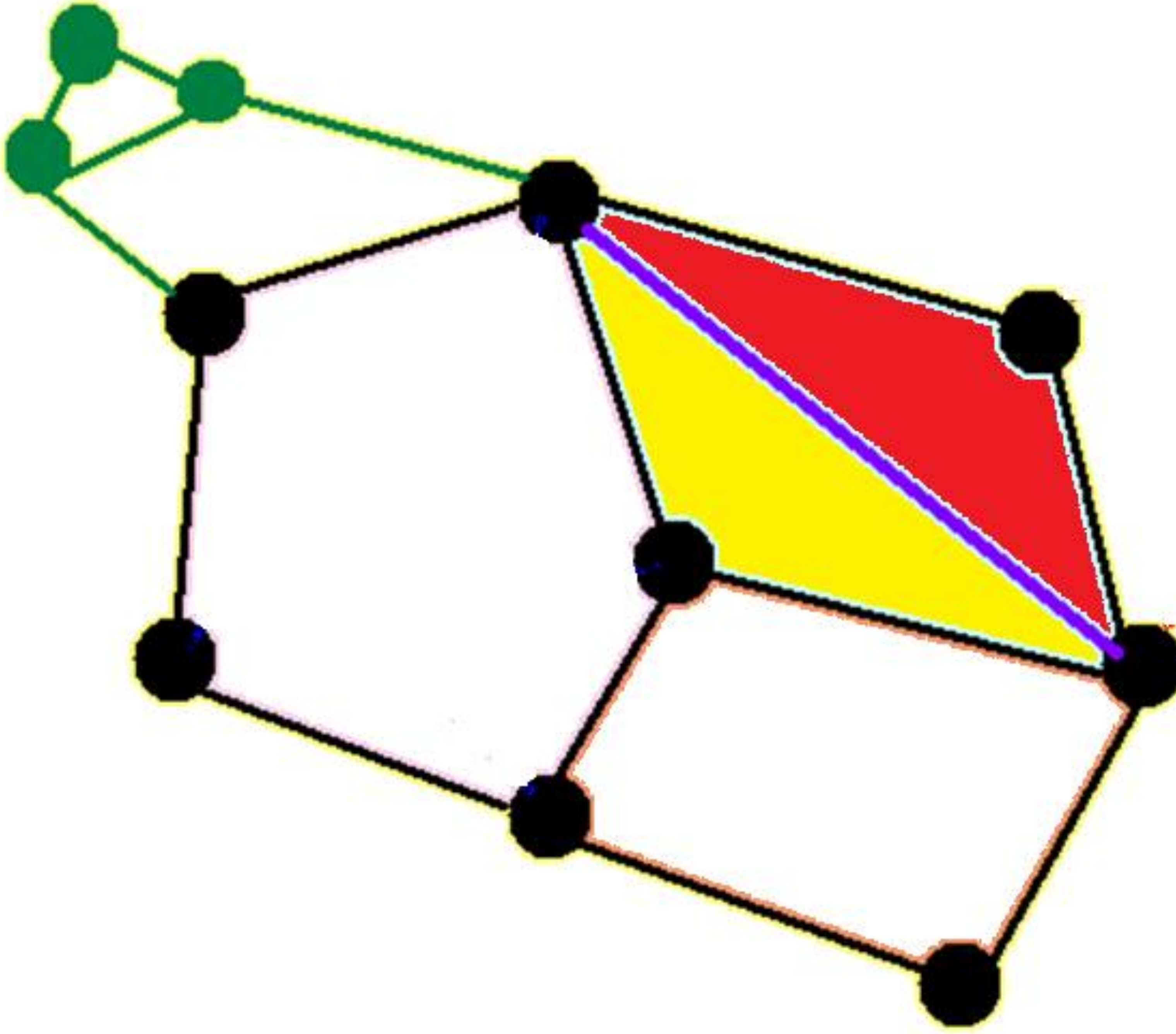


[Basis]
The connected graphs on n vertices with a minimum number of edges are trees.
If T is a tree, then it has n-1 edges and one face when embedded in the plane.
Checking the formula:
$1 = (n-1) - n + 2 \Longrightarrow 1 = 1$ so the base case holds.

[Induction step (m → m+1)]

Assume that for a planar embedding $\widetilde{G}$ of a connected planar graph G with n vertices and m edges that f= m-n+2.
We want to prove that adding one edge (while maintaining planarity) gives a new planar embedding $\widetilde{H}$ of a graph H such that f' (the number of faces of H)
satisfies f' = m' – n + 2
where m'= m+1 is the number of edges of H.





Adding one edge adds one more face.

Therefore, $f' = f + 1$. Recall $m' = m+1$.

Checking the formula:

$f' = m' - n + 2$

means that

$f+1 = m+1 - n + 2$

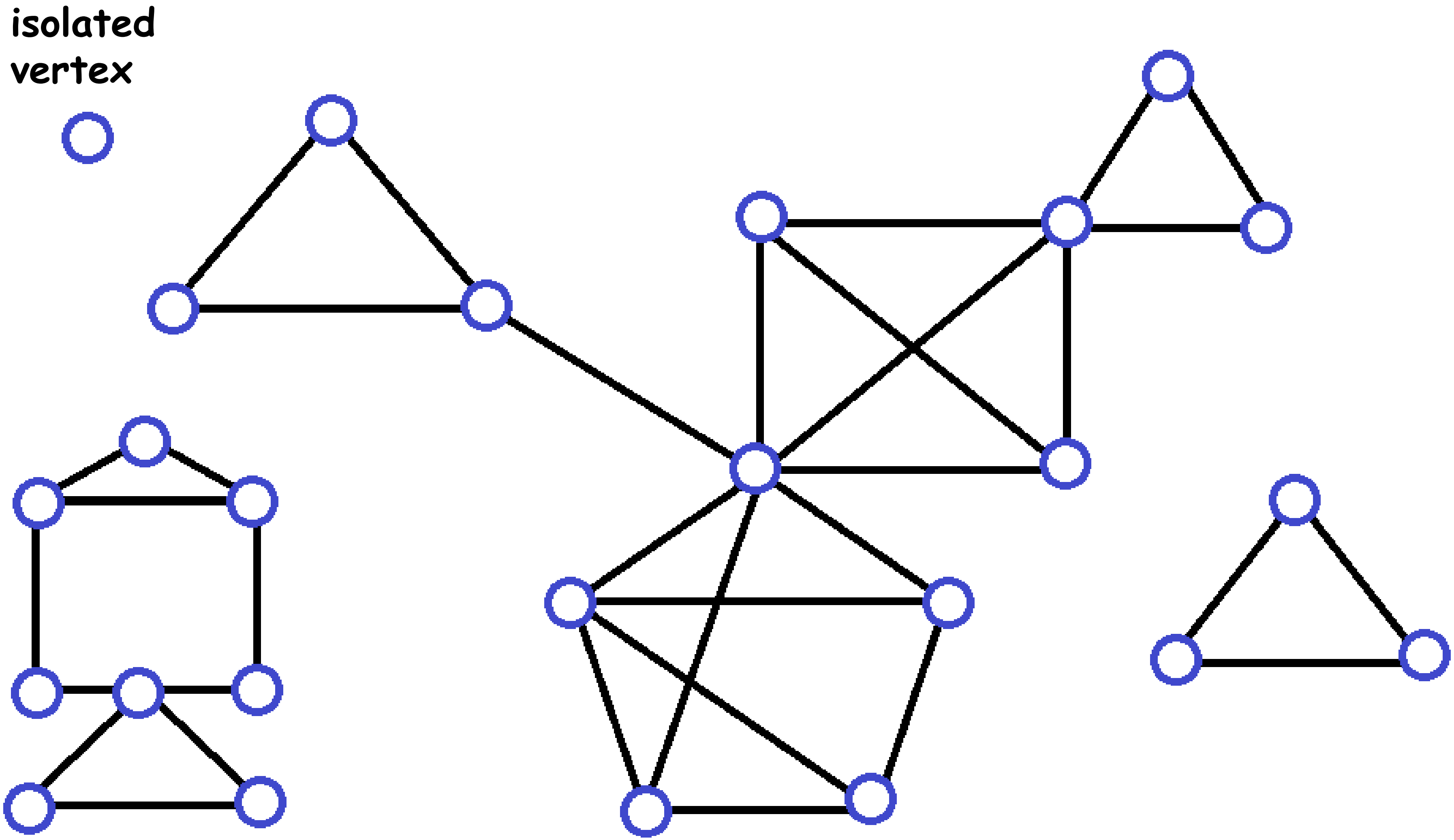
subtracting one from both sides gives

$f = m - n + 2$ which we know is true by induction.

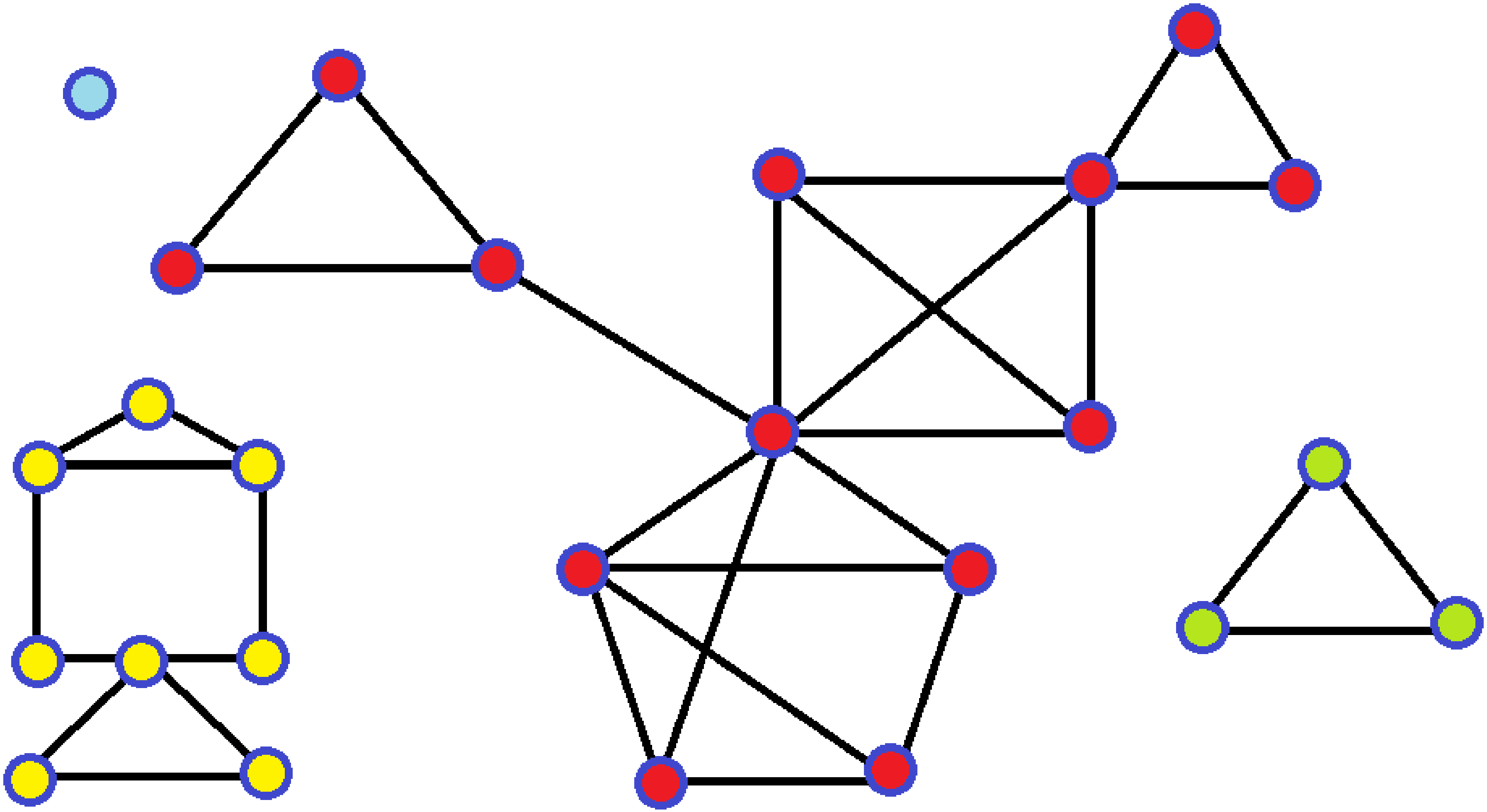
Pre-processing for an embedding algorithm.

1. Break graph into its connected components.
2. For each connected component, break it into its 2-connected components (maximal subgraphs having no cut vertex).

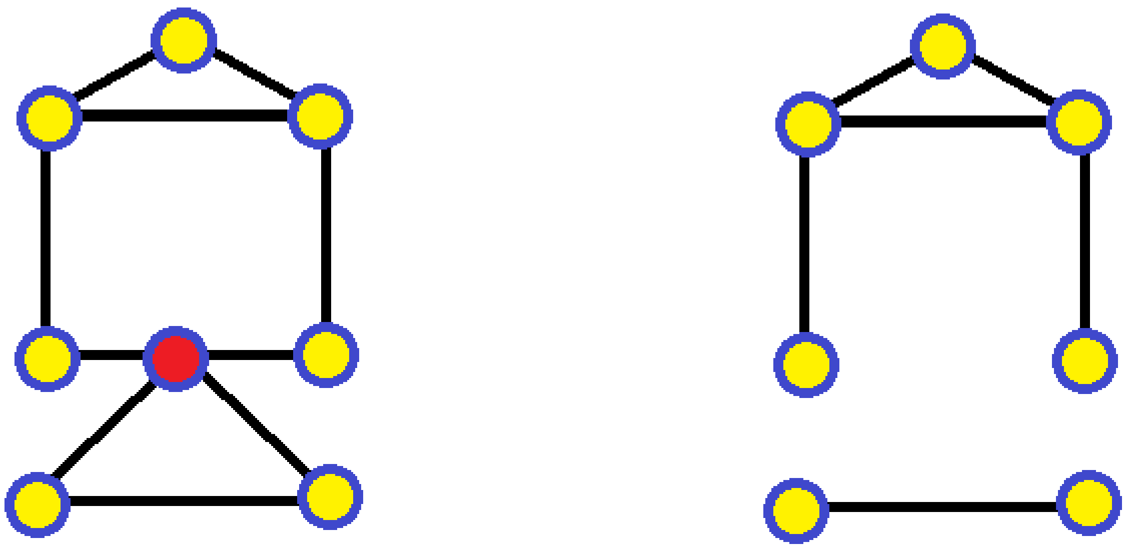
A disconnected graph:



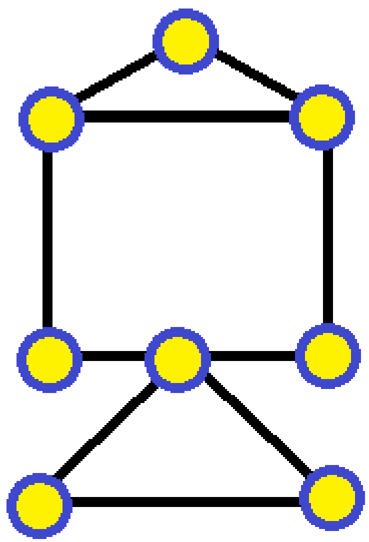
First split into its 4 connected components:

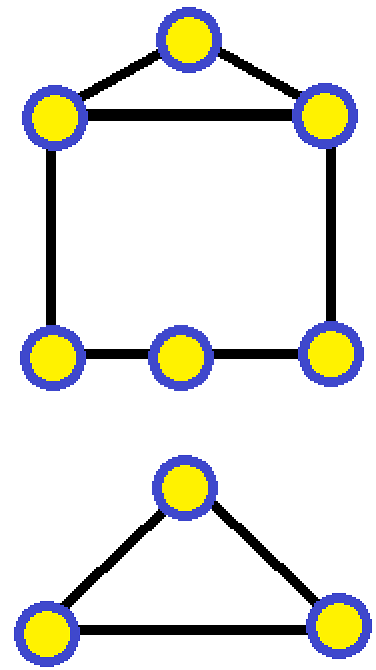


The yellow component has a cut vertex:

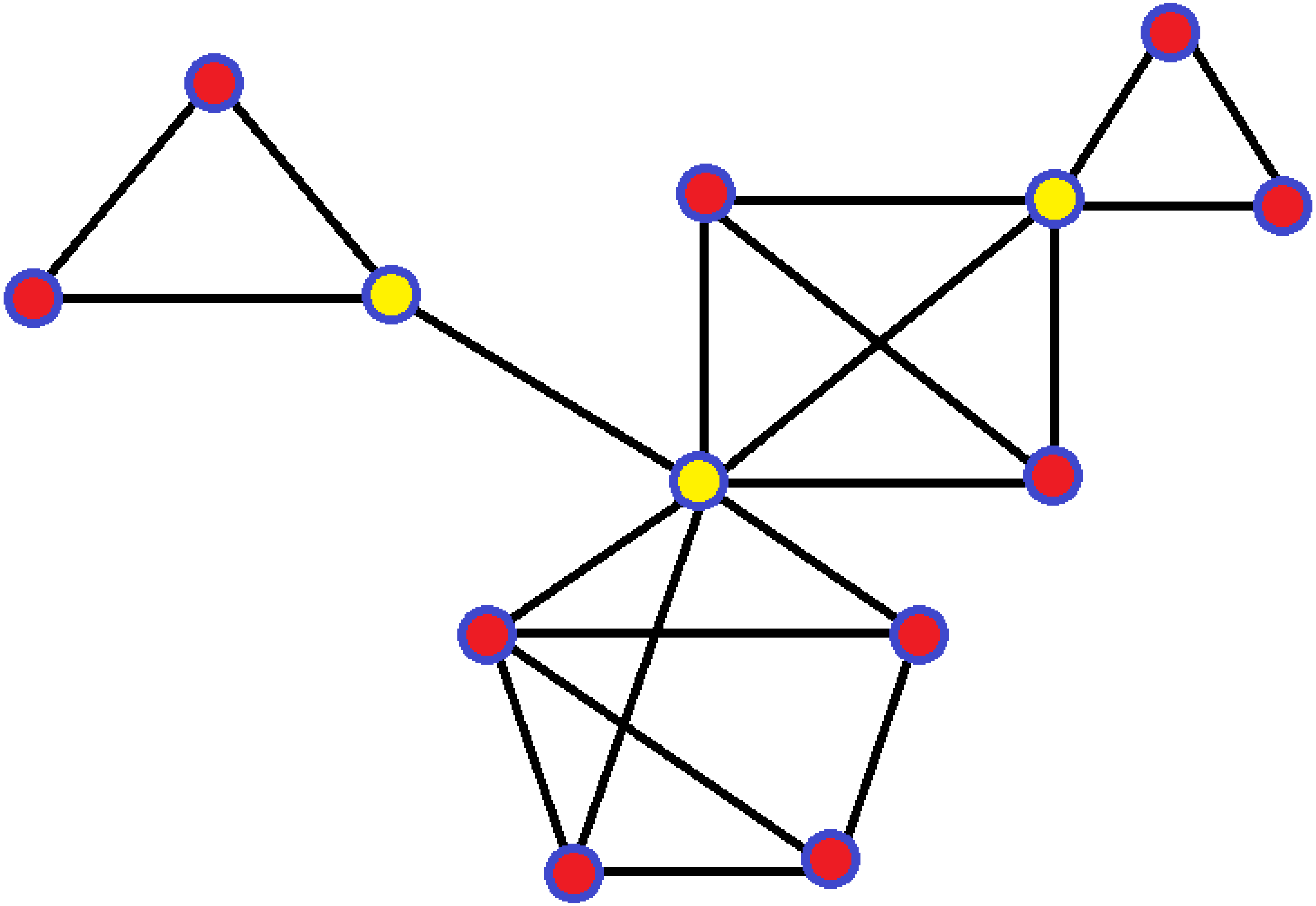


The 2-connected components of the yellow component:

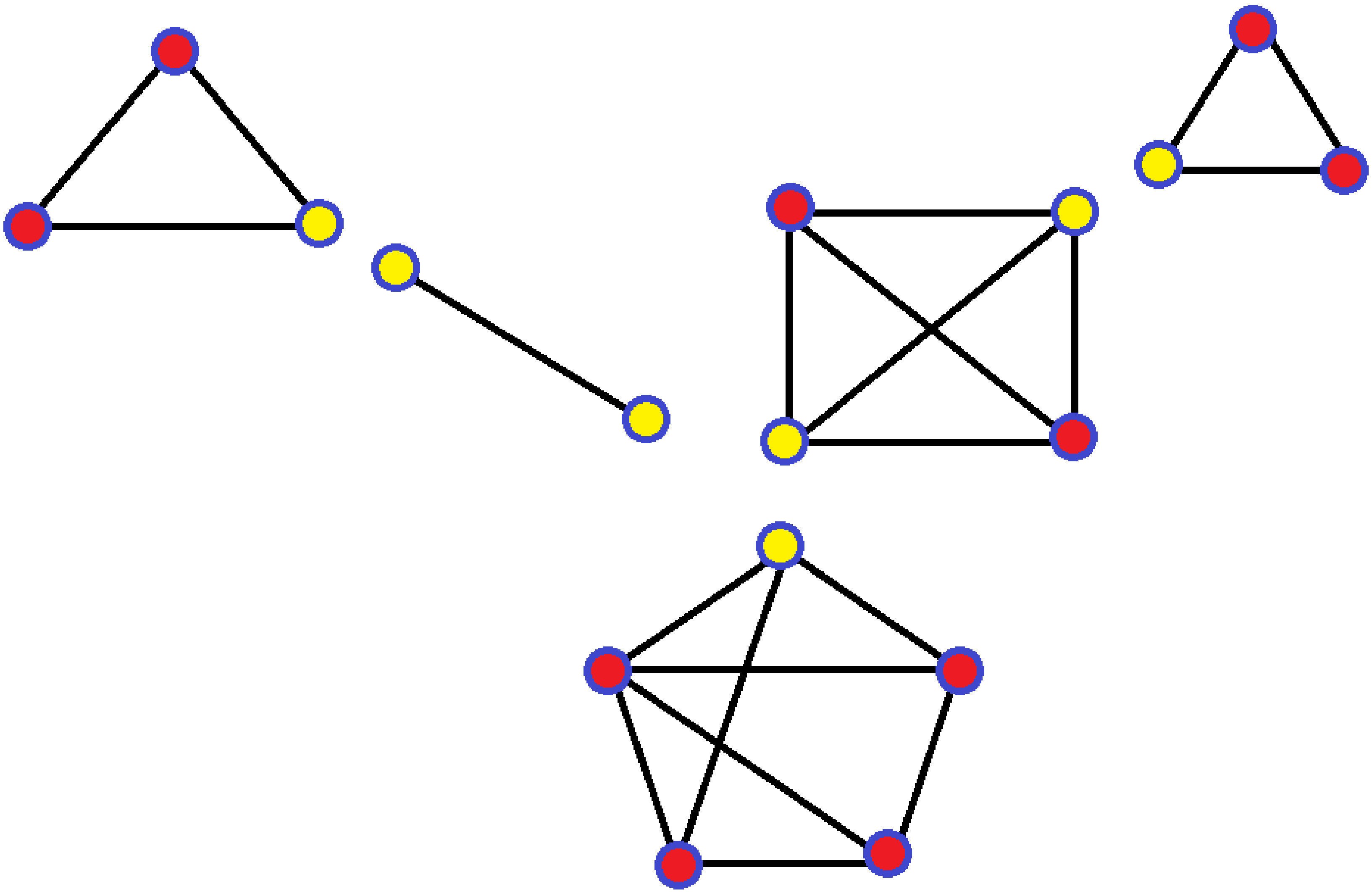


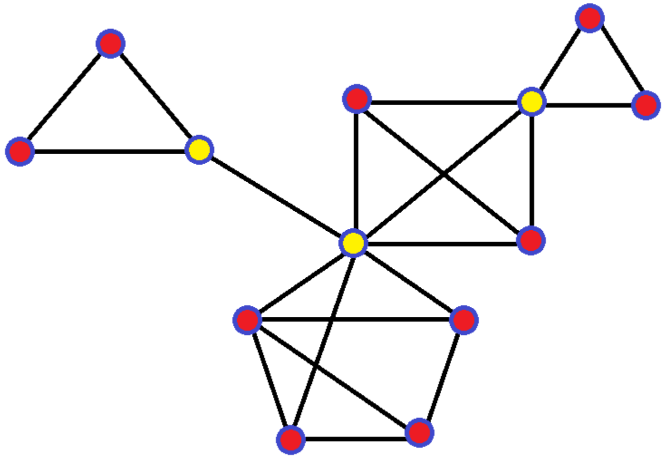


The red component: the yellow vertices are cut vertices.

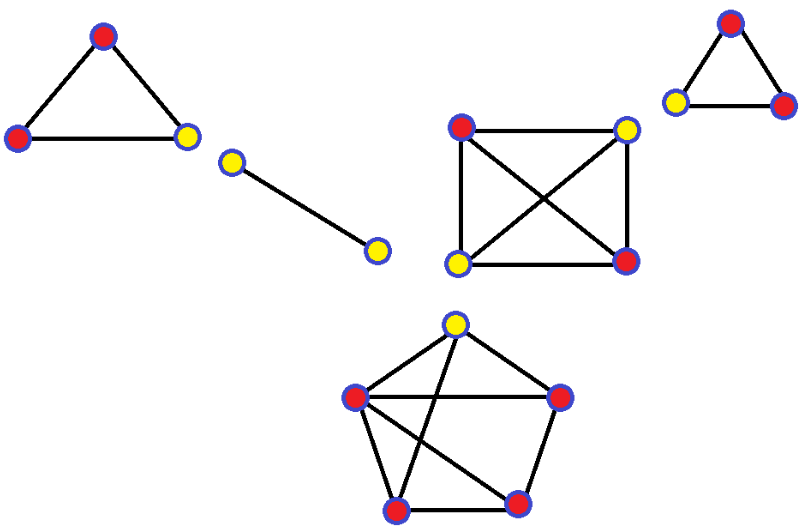


The 2-connected components of the red component:





How do we decompose the graph like this using a computer algorithm?



The easiest way:

BFS (Breadth First Search)

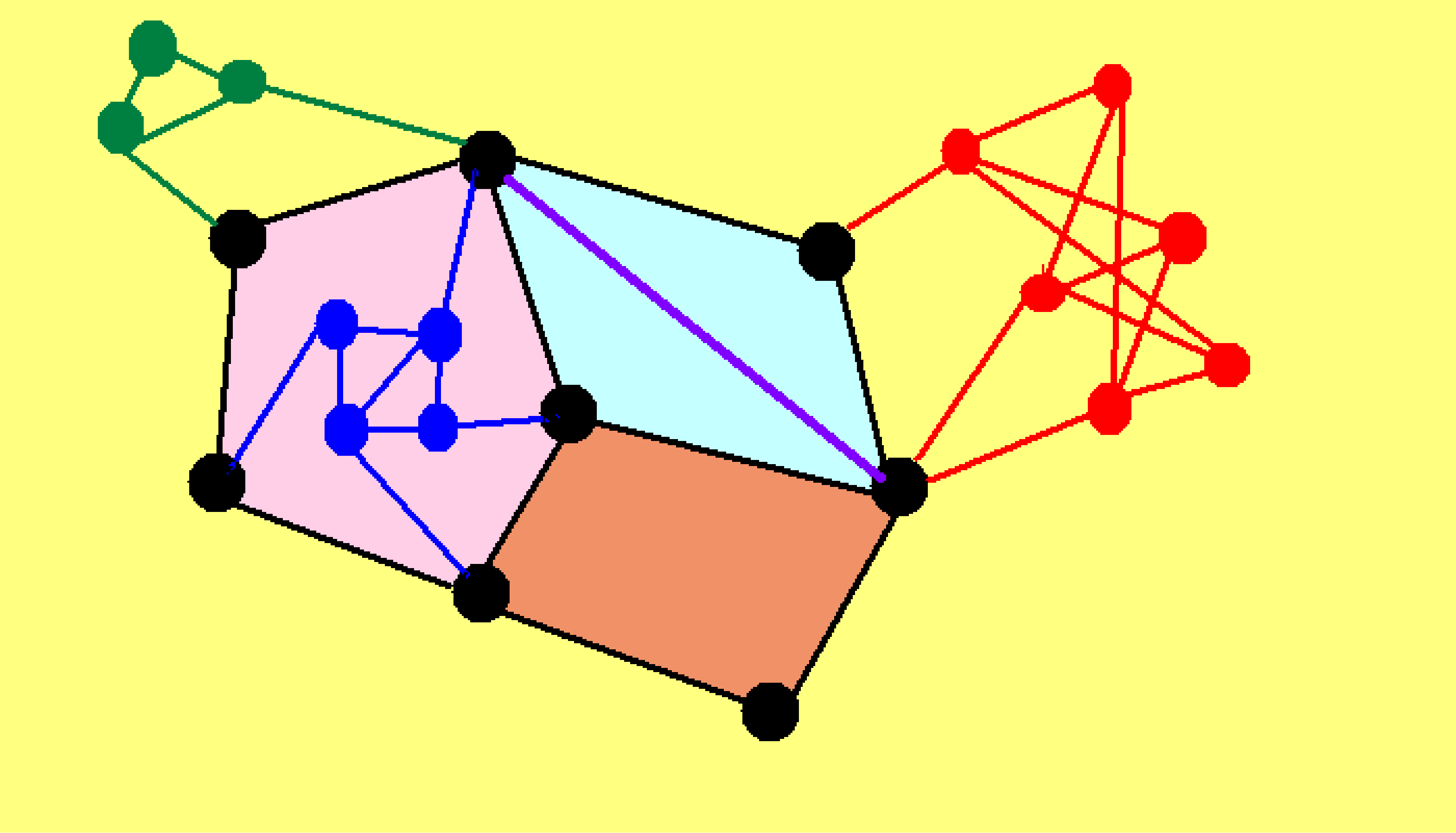
A bridge with respect to a subgraph H of a graph G is either:

1. An edge e=(u, v) which is not in H but both u and v are in H.
2. A connected component C of G-H plus any edges that are incident to one vertex in C and one vertex in H plus the endpoints of these edges.

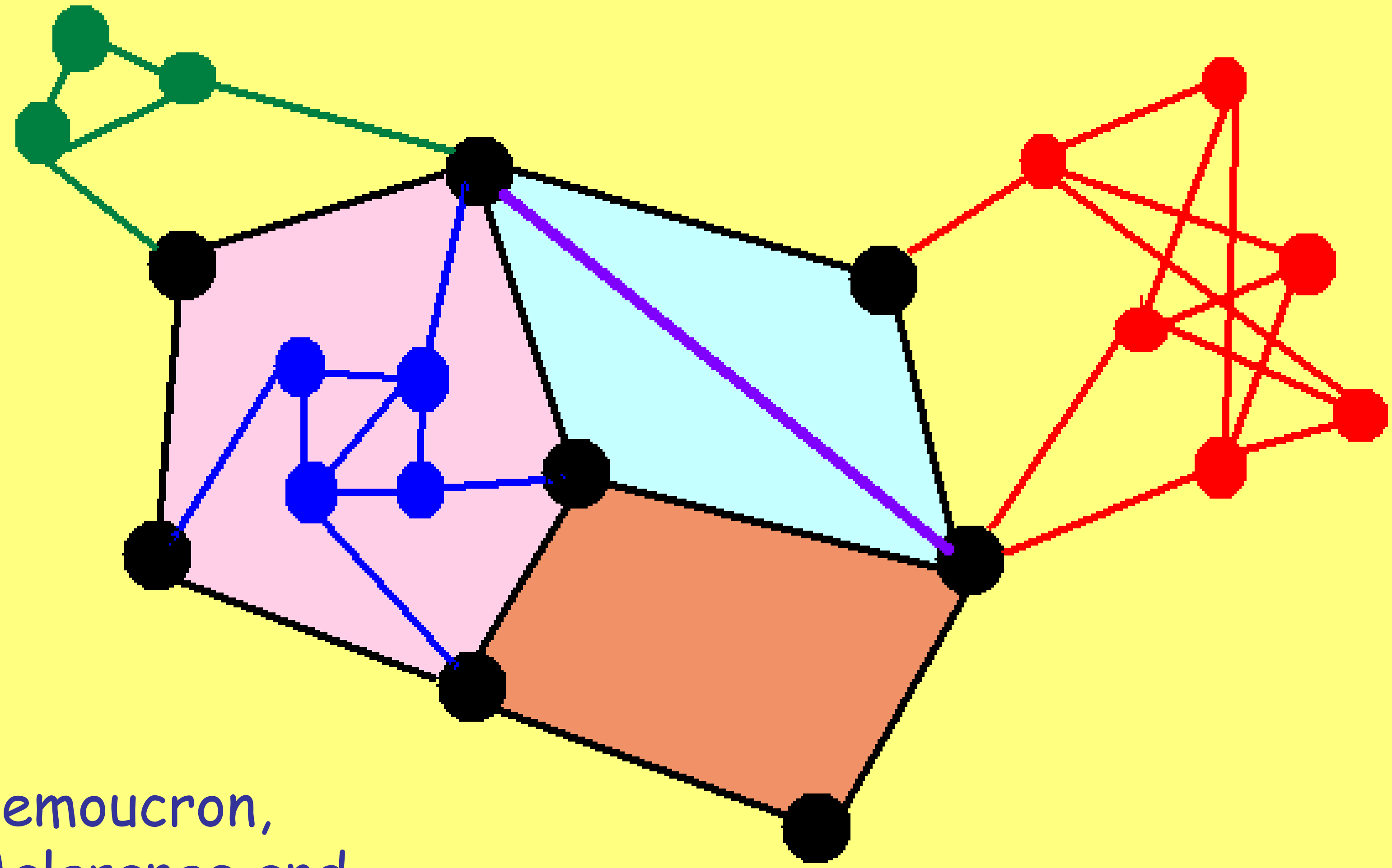
How can you find the bridges with respect to a cut vertex v?

How can we find a planar embedding of each 2-connected component of a graph?
One simple solution: Algorithm by Demoucron, Malgrange and Pertuiset.

```
@ARTICLE{genus:DMP,
  AUTHOR = {G. Demoucron and Y. Malgrange
                   and R. Pertuiset},
  TITLE = {Graphes Planaires},
  JOURNAL = {Rev. Fran\c{c}aise Recherche
                   Op\'{e}rationnelle},
  YEAR = {1964},
  VOLUME = {8},
  PAGES = {33--47}   }
```

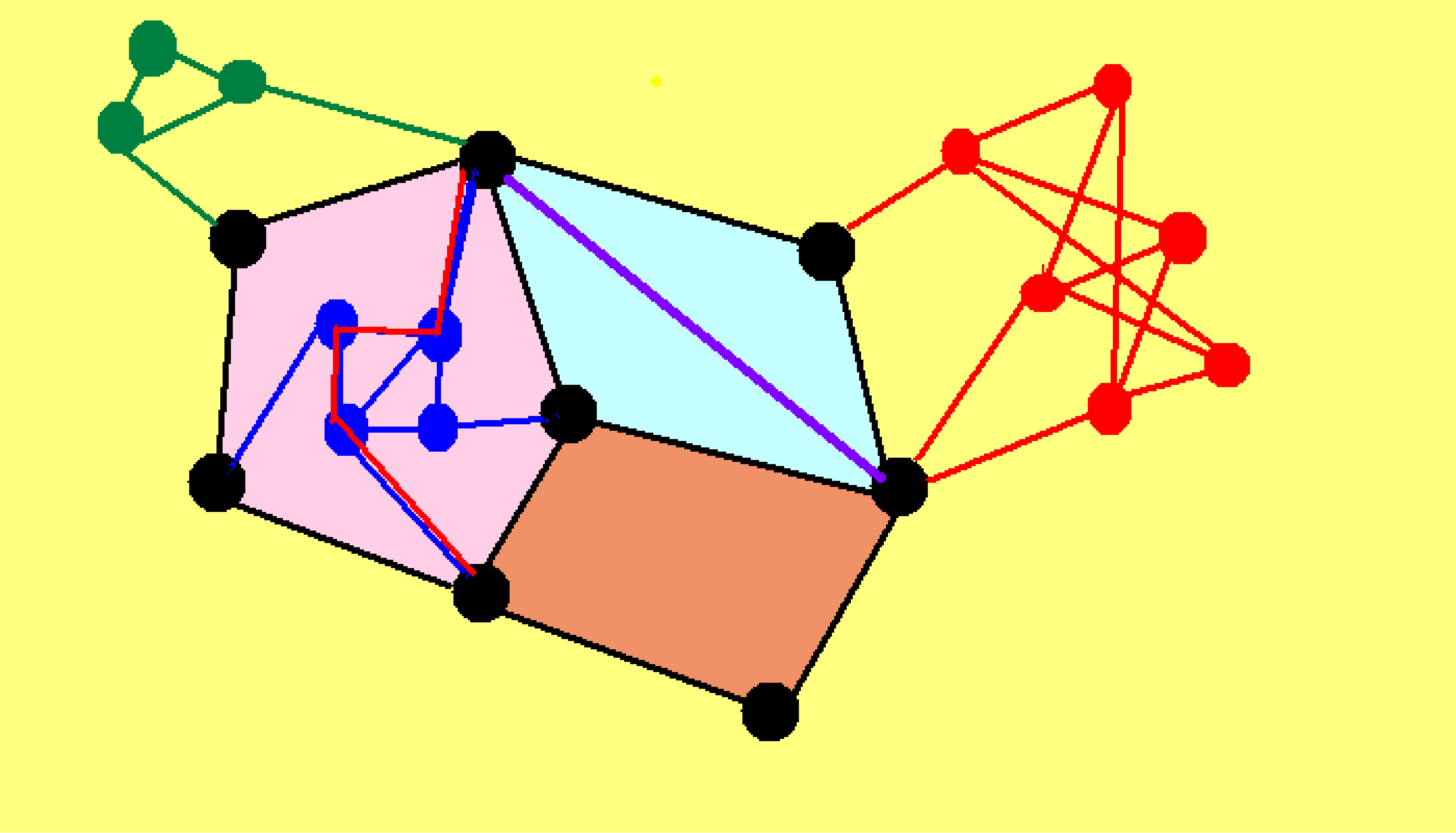


A bridge can be drawn in a face if all its points of attachment lie on that face.

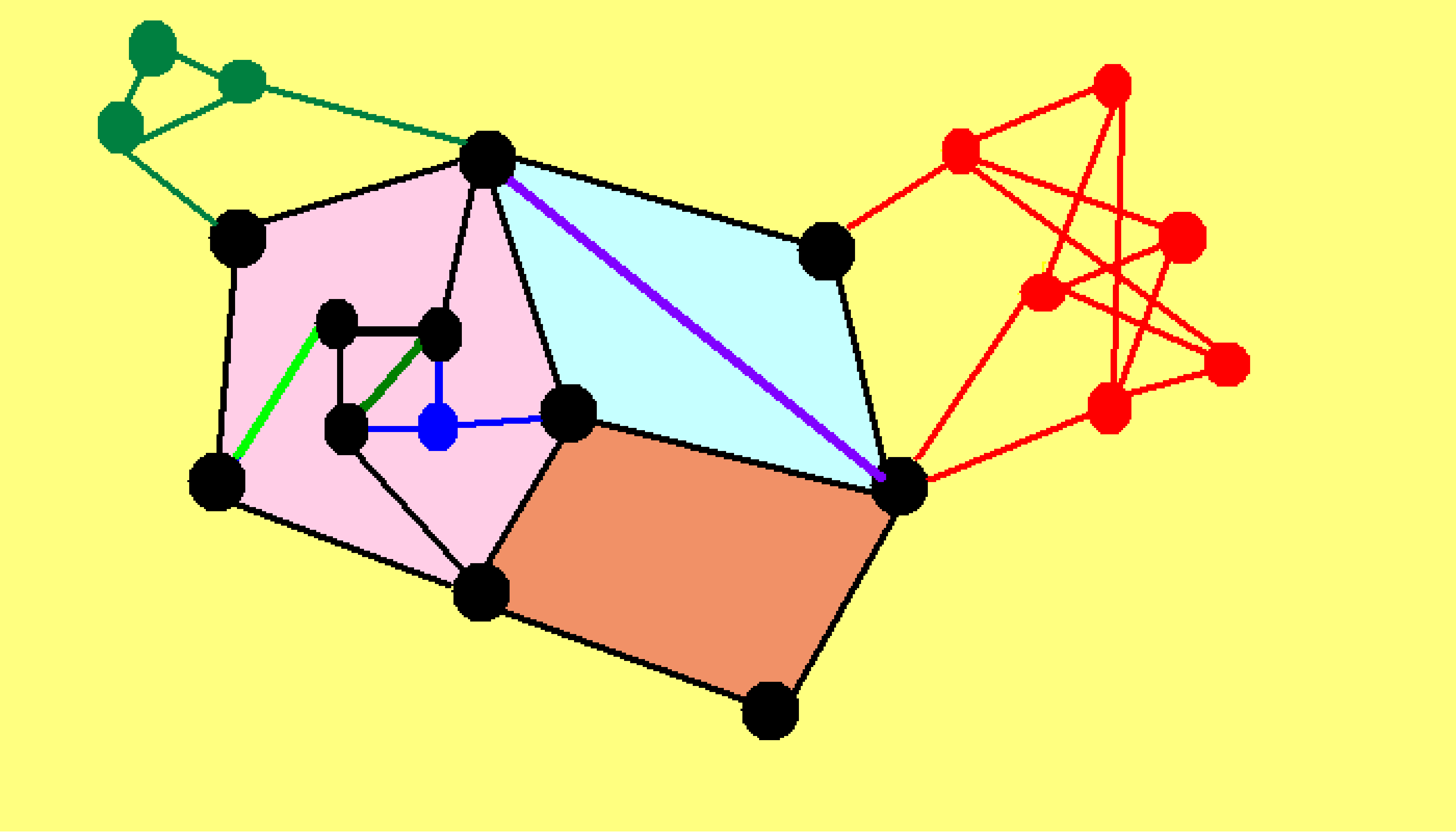


Democron,
Malgrange and
Pertuiset '64:

1. Find a bridge which can be drawn in a minimum number of faces (the blue bridge).



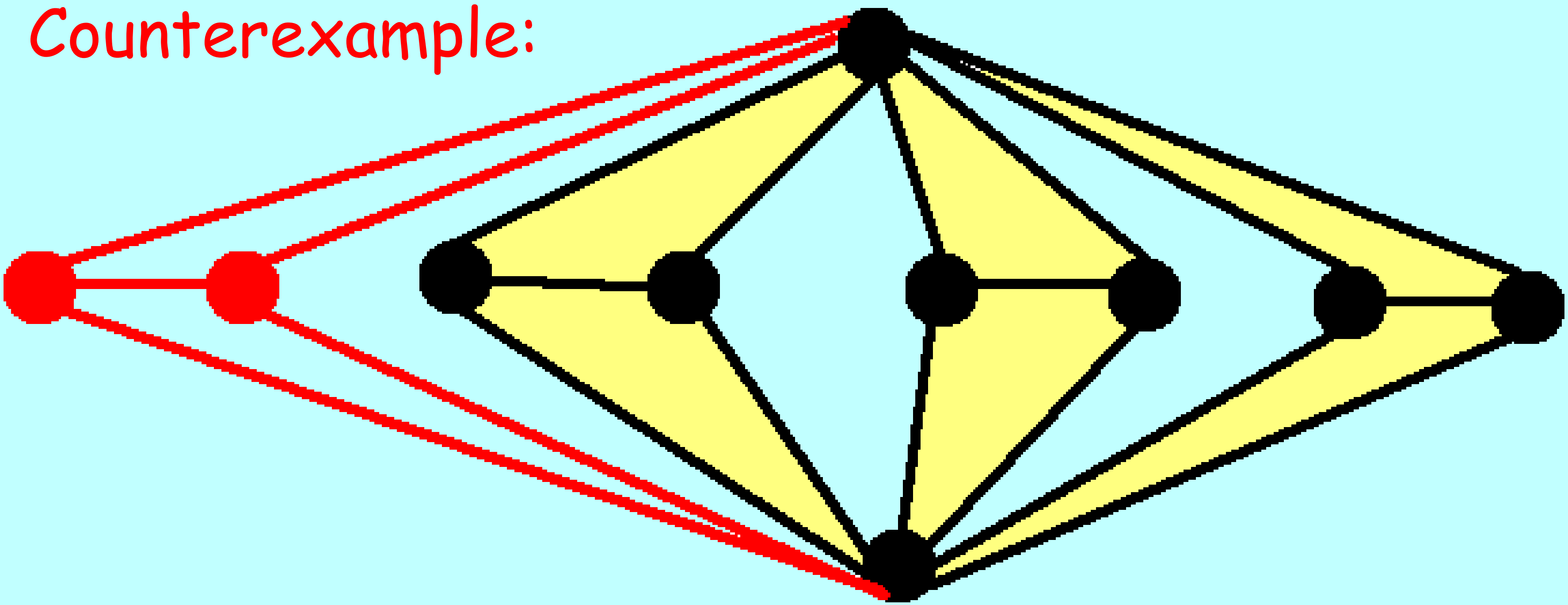
2. Find a path between two points of attachment for that bridge and add the path to the embedding.



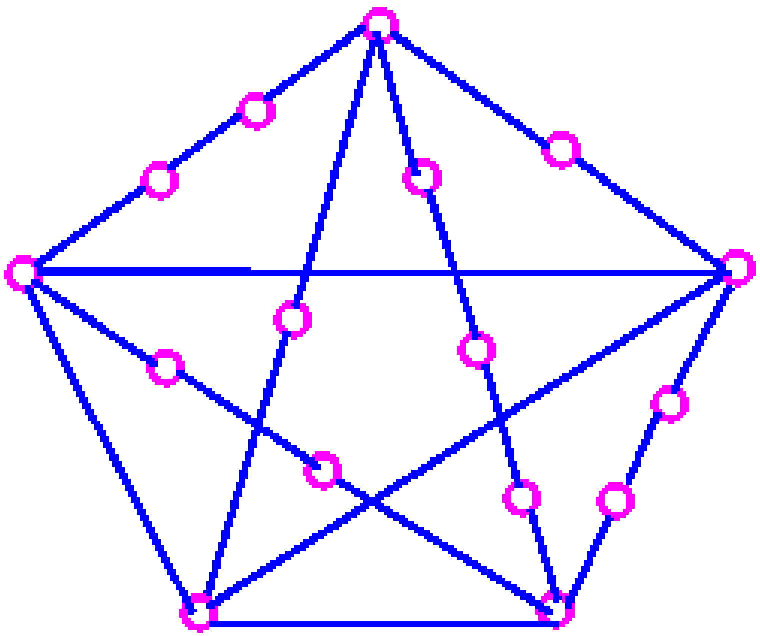
No backtracking required for planarity testing!

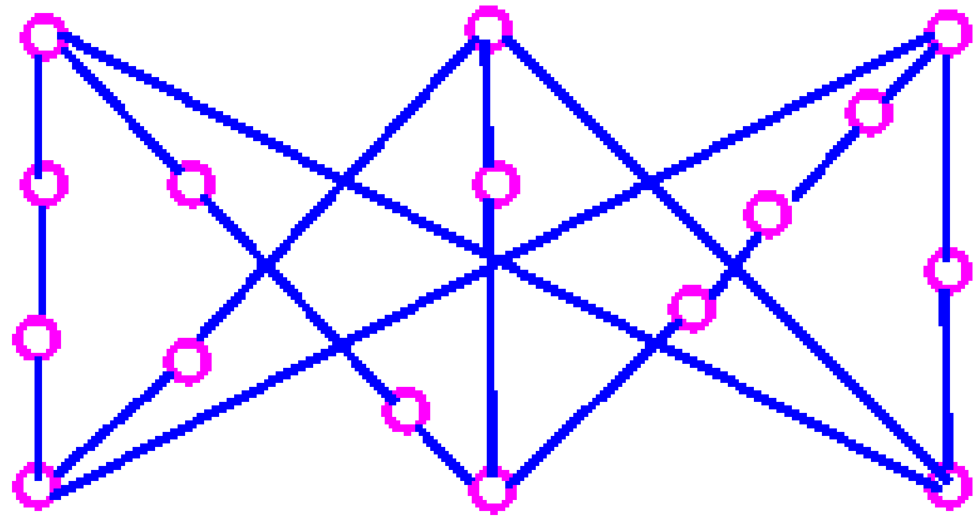
Gibbons: if *G* is 2-vertex connected, every bridge of *G* has at least two points of contact and can therefore be drawn in just two faces.

Counterexample:



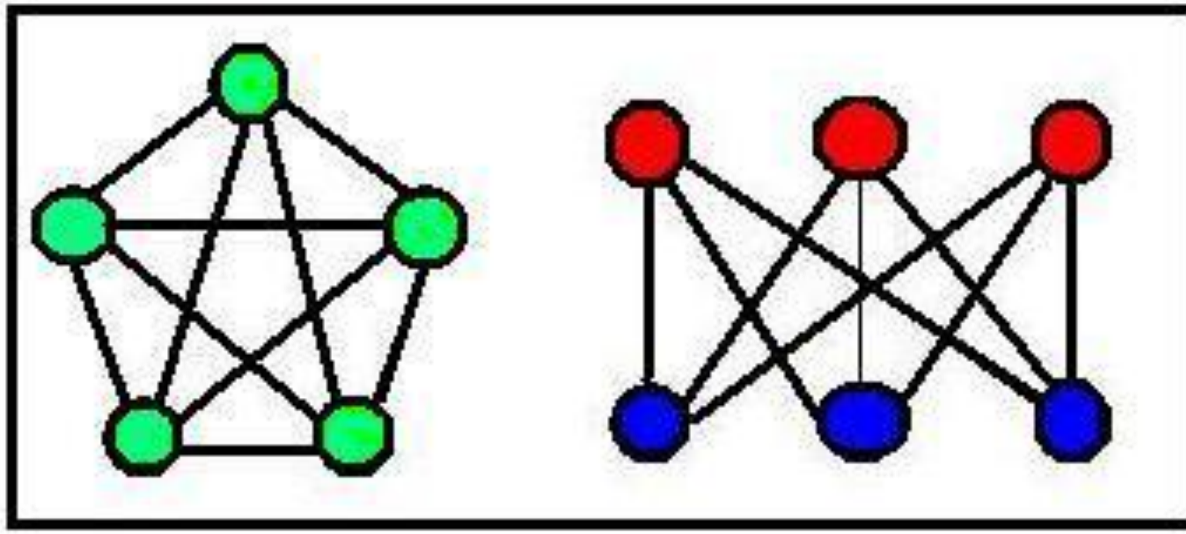
Graphs homeomorphic to K_5 and $K_{3,3}$:





Rashid Bin
Muhammad

Kuratowski's theorem: If G is not planar then it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.



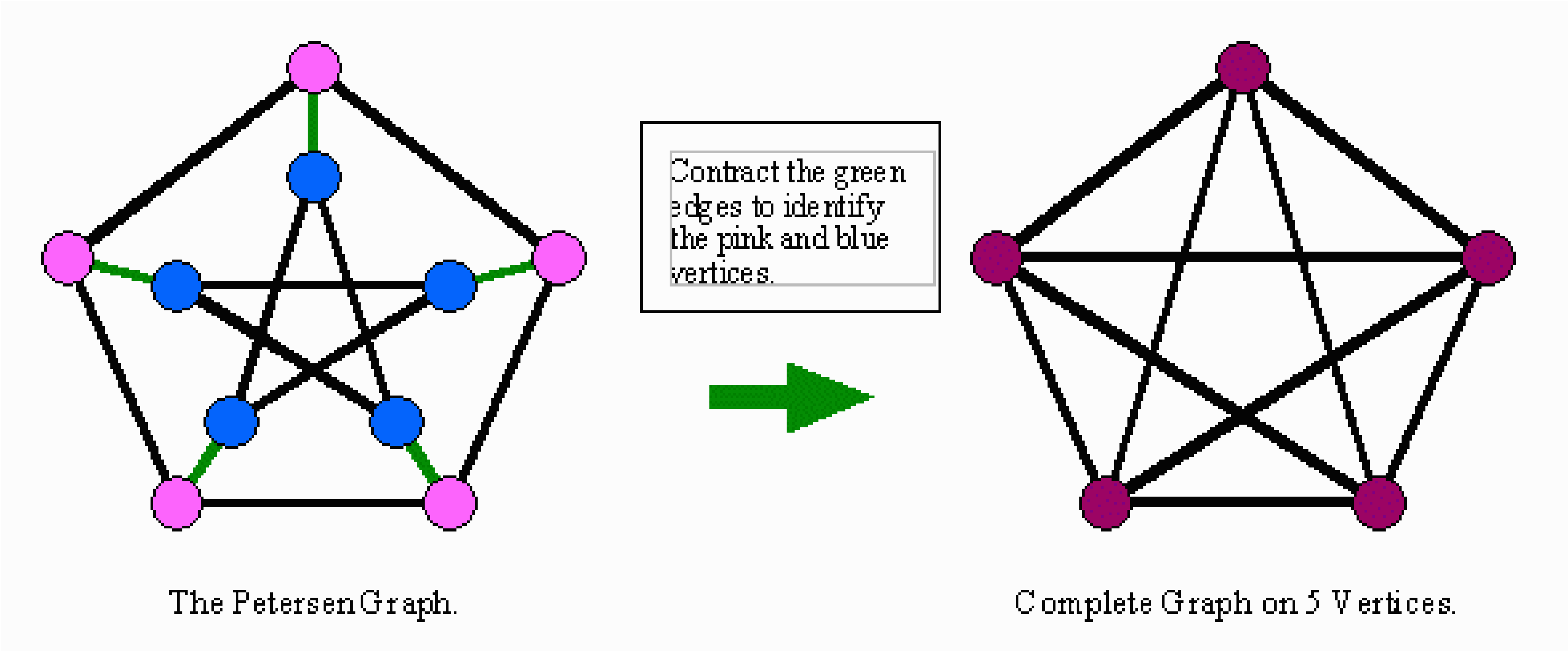
Topological obstruction for surface S:

degrees ≥3,does not embed on S,

G-e embeds on S for all e.

Minor Order Obstruction: Topological obstruction and $G \cdot e$ embeds on S for all e.

Wagner's theorem: G is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor.



Dale Winter

Obstructions for Surfaces

Fact: for any orientable or non-orientable surface, the set of obstructions is finite.

Consequence of Robertson & Seymour theory but also proved independently:

Orientable surfaces: Bodendiek & Wagner, '89

Non-orientable: Archdeacon & Huneke, '89.

How many torus obstructions are there?

Minor Order Torus Obstructions: 1754

8 :	3
9 :	43
10 :	457
11 :	2839
12 :	6426
13 :	5394
14 :	1838
15 :	291
16 :	54
17 :	8
18 :	1

n/m:	18	19	20	21	22	23	24	25	26	27	28	29	30
8 :	0	0	0	0	1	0	1	1	0	0	0	0	0
9 :	0	2	5	2	9	13	6	2	4	0	0	0	0
10 :	0	15	3	18	31	117	90	92	72	17	1	0	1
11 :	5	2	0	46	131	569	998	745	287	44	8	3	1
12 :	1	0	0	52	238	1218	2517	1827	472	79	21	1	0
13 :	0	0	0	5	98	836	1985	1907	455	65	43	0	0
14 :	0	0	0	0	9	68	463	942	222	41	92	1	0
15 :	0	0	0	0	0	0	21	118	43	13	91	5	0
16 :	0	0	0	0	0	0	0	4	3	5	41	0	1
17 :	0	0	0	0	0	0	0	0	0	0	8	0	0
18 :	0	0	0	0	0	0	0	0	0	0	1	0	0

All Torus Obstructions Found So Far:

n/m:	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
8 :	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
9 :	0	2	5	2	9	17	6	2	5	0	0	0	0	0	0	0	0	0	0
10 :	0	15	9	35	40	190	170	102	76	21	1	0	1	0	0	0	0	0	0
11 :	5	2	49	87	270	892	1878	1092	501	124	22	4	1	0	0	0	0	0	0
12 :	1	12	6	201	808	2698	6688	6372	1933	482	94	6	2	0	0	0	0	0	0
13 :	0	0	12	19	820	4967	12781	16704	7182	1476	266	52	1	0	0	0	0	0	0
14 :	0	0	0	9	38	2476	15219	24352	16298	3858	808	215	19	0	0	0	0	0	0
15 :	0	0	0	0	0	33	3646	22402	20954	8378	1859	708	184	5	0	0	0	0	0
16 :	0	0	0	0	0	0	20	2689	17469	10578	3077	1282	694	66	1	0	0	0	0
17 :	0	0	0	0	0	0	0	0	837	8099	4152	1090	1059	368	11	0	0	0	0
18 :	0	0	0	0	0	0	0	0	0	133	2332	1471	511	639	102	1	0	0	0
19 :	0	0	0	0	0	0	0	0	0	0	0	393	435	292	255	15	0	0	0
20 :	0	0	0	0	0	0	0	0	0	0	0	0	39	100	164	63	2	0	0
21 :	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	63	1	0	0
22 :	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	22	0	0
23 :	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0
24 :	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2