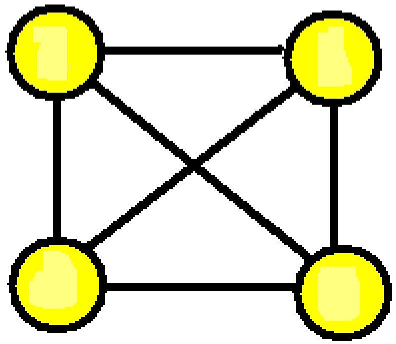
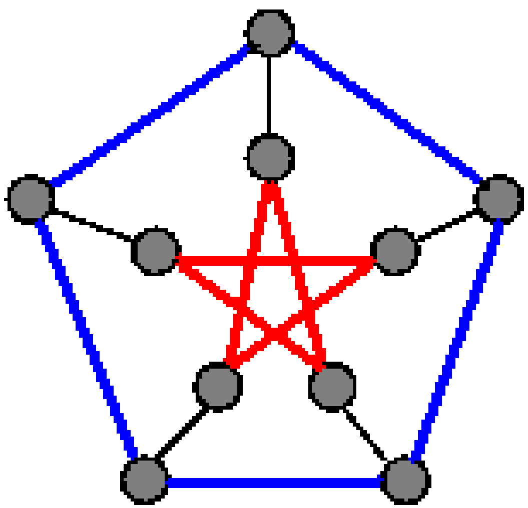
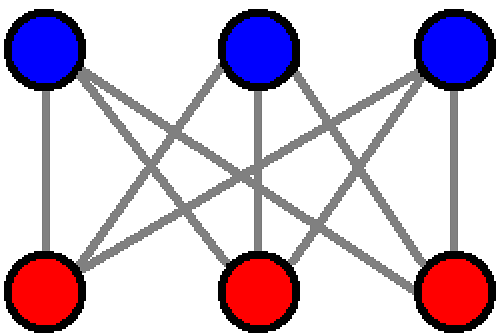
Fast Backtracking Principles applied to find new Cages



Brendan McKay
Wendy Myrvold
Jacqueline Nadon
plus some new work
with
Jenni Woodcock
and Robert Aftias

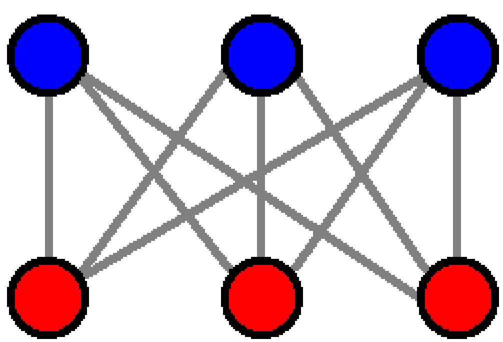




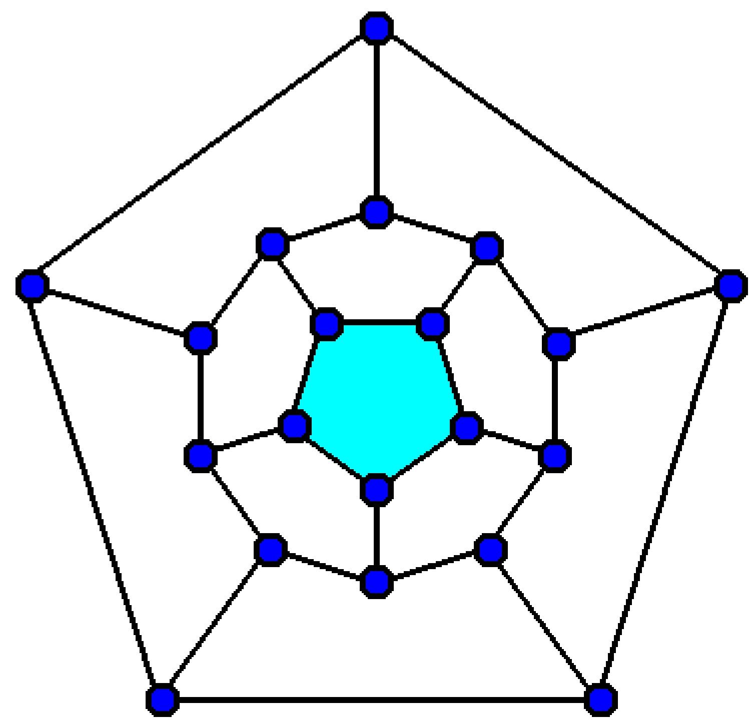
girth: size of a smallest cycle

r-regular: every vertex has degree r.

(r,g)-cage: r-regular graph of girth g with a minimum number of vertices

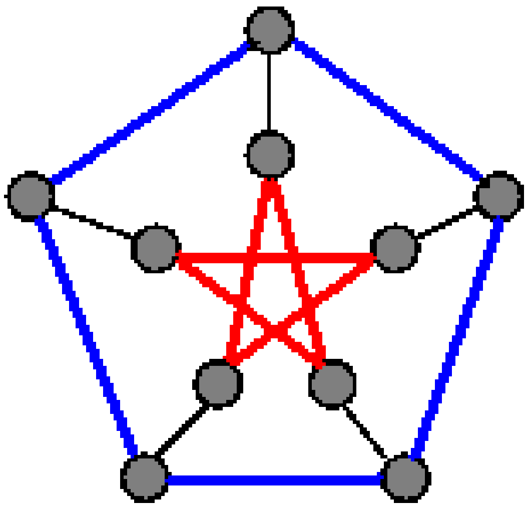


Girth 4

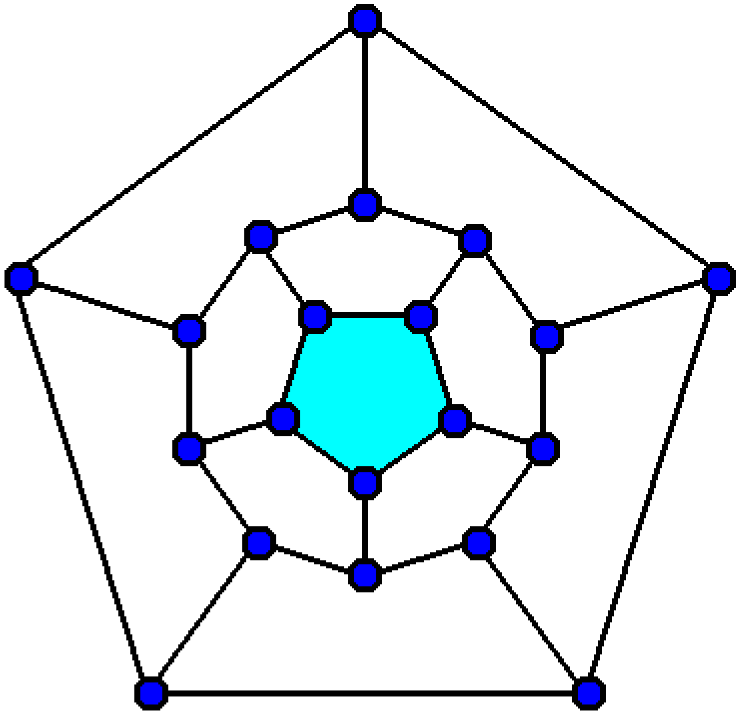


Girth 5

(r,g)-cage: r-regular graph of girth g with a minimum number of vertices

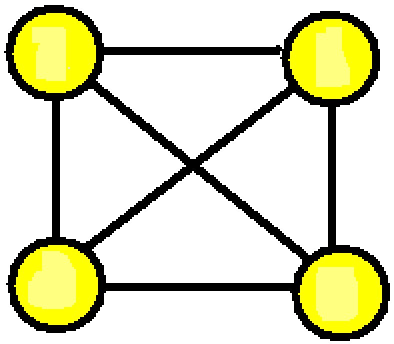


(3,5)-cage

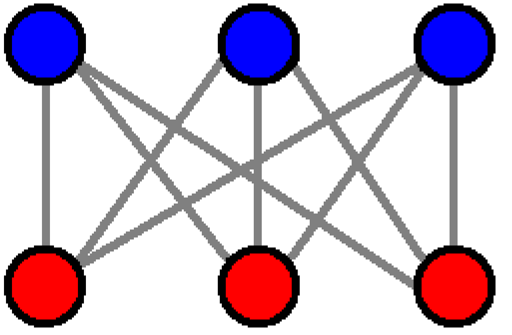


Not a (3,5)-cage

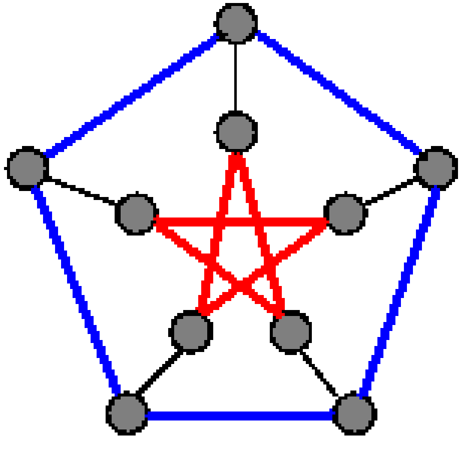
Some cages:



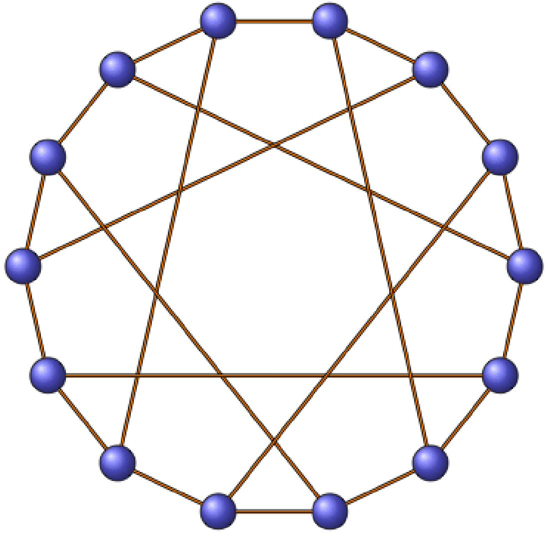
(3,3)-cage: K_4



(3,4)-cage: $K_{3,3}$

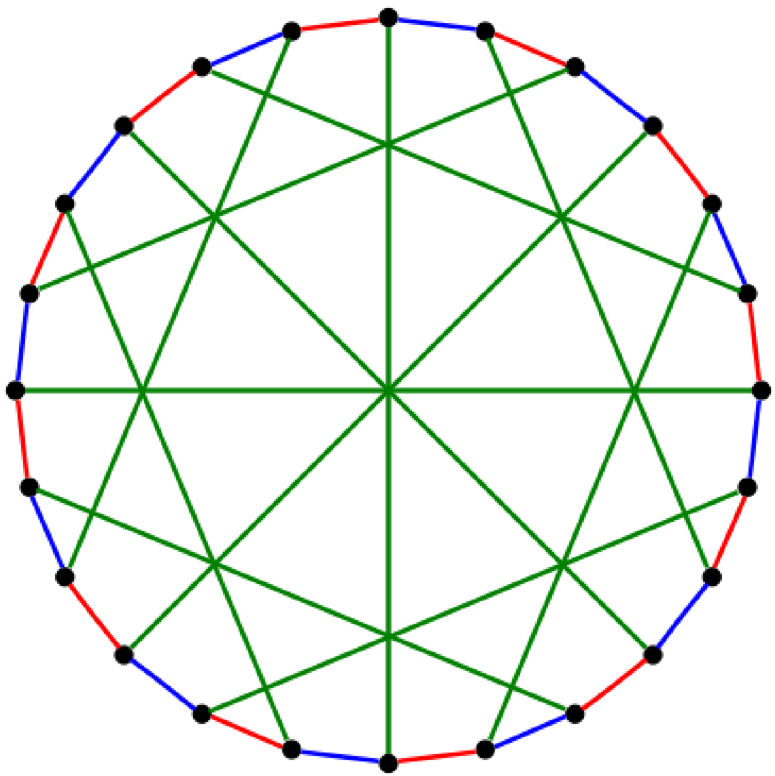


(3,5)-cage:
Petersen graph



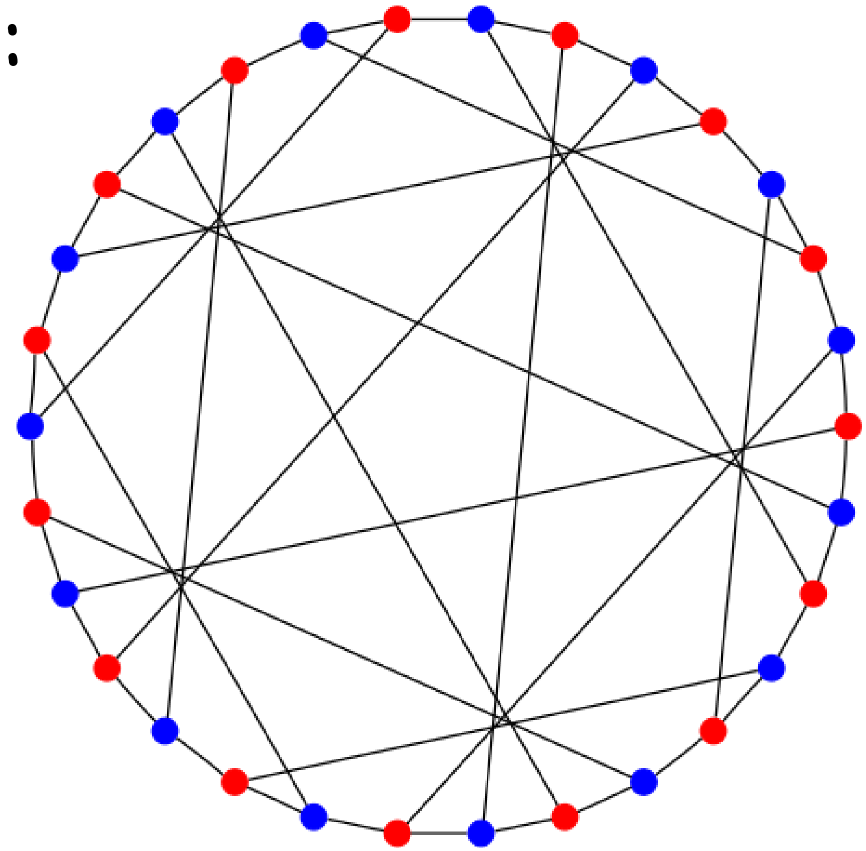
(3,6)-cage:
Heawood graph

More 3-regular cages:



(3,7)-cage:

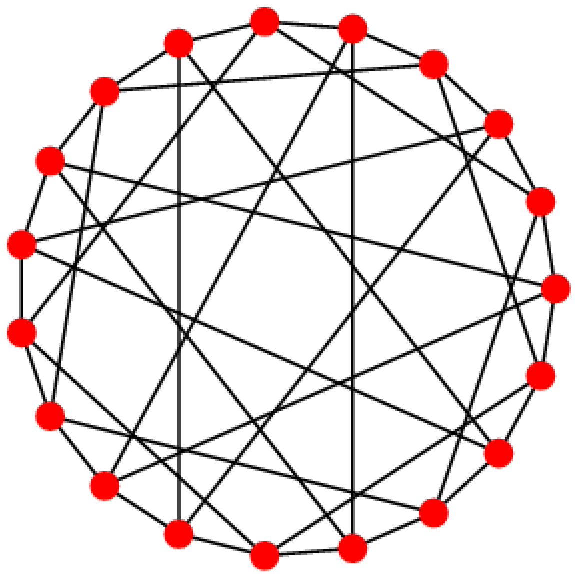
McGee graph



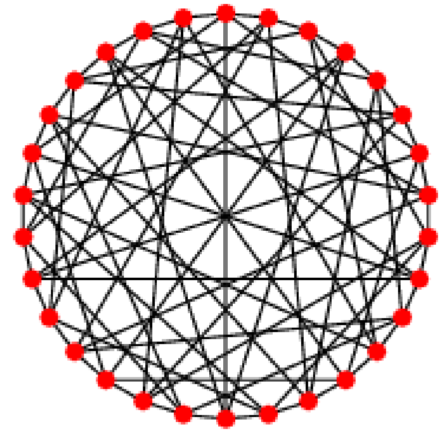
(3,8)-cage:

Tutte-Coxeter graph

Some other cages

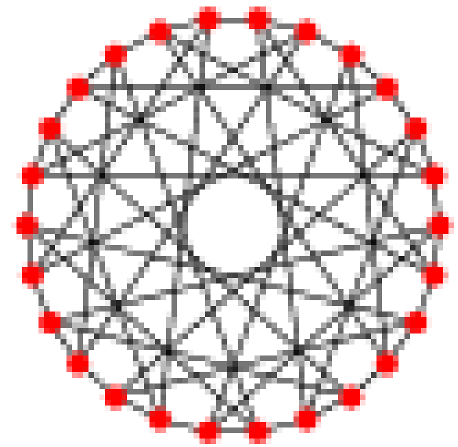


(4,5)-cage:
Robertson graph

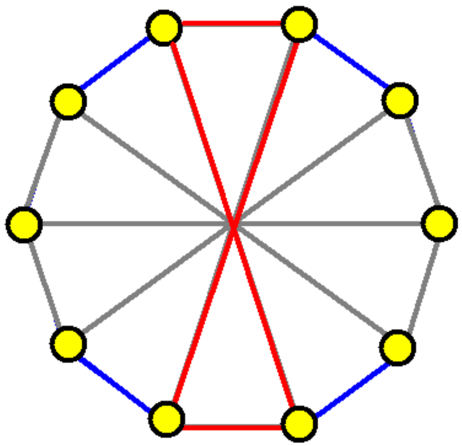


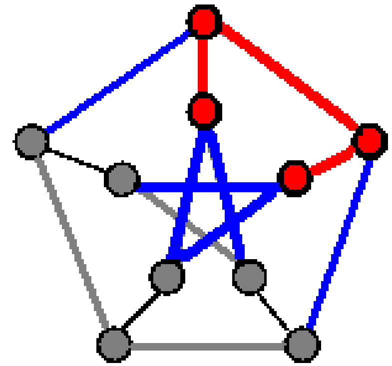
(5,5)-cage: Robertson-Wagner graph

(4,6)–cage



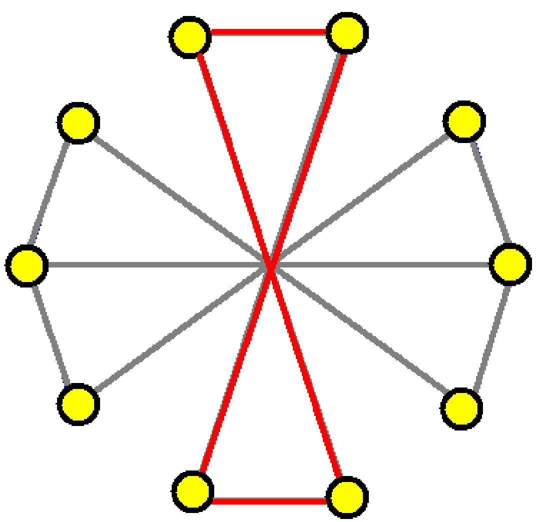
Motivation: Network Reliability

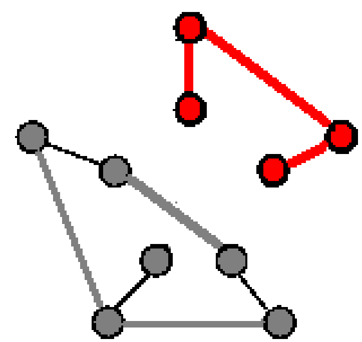


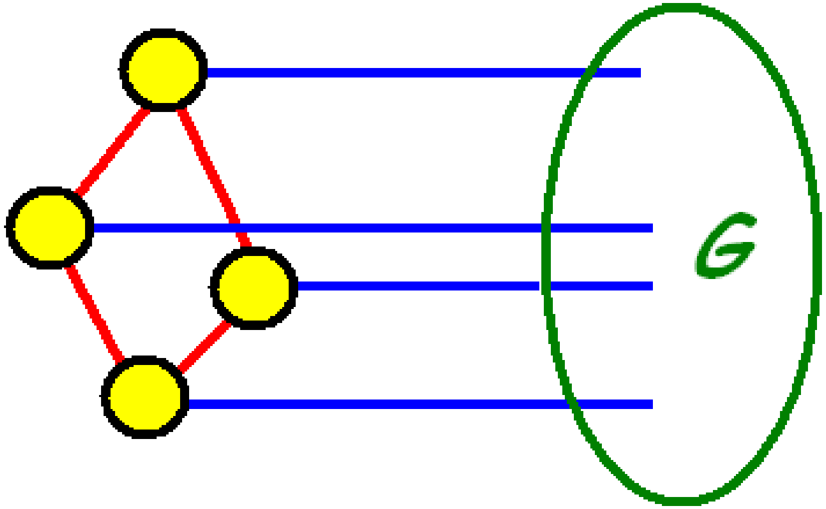


6-edge cut of Petersen graph

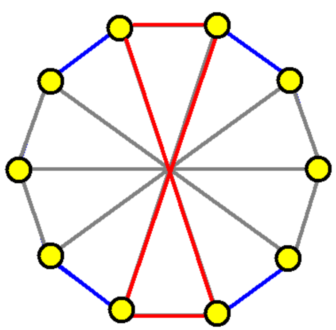
4-edge cut of circulant

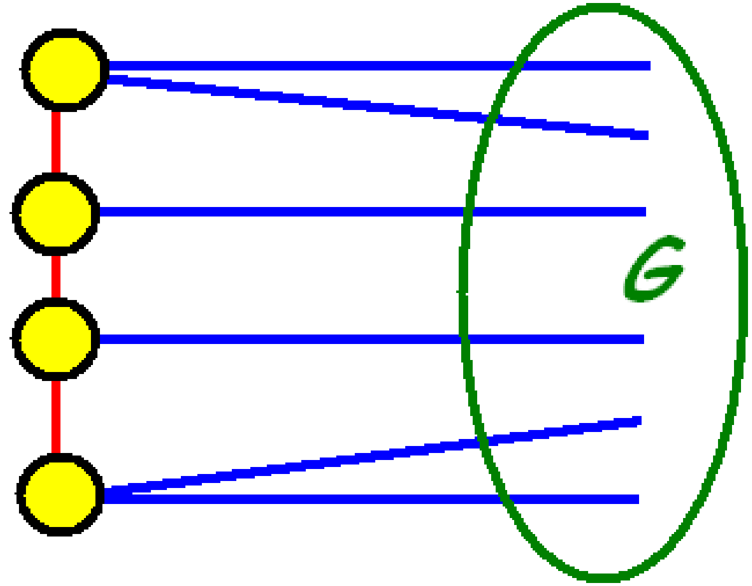




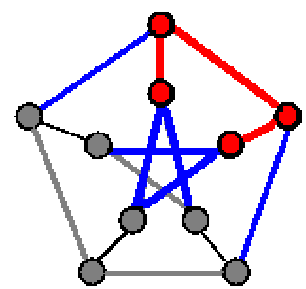


Girth 4





Girth > 4

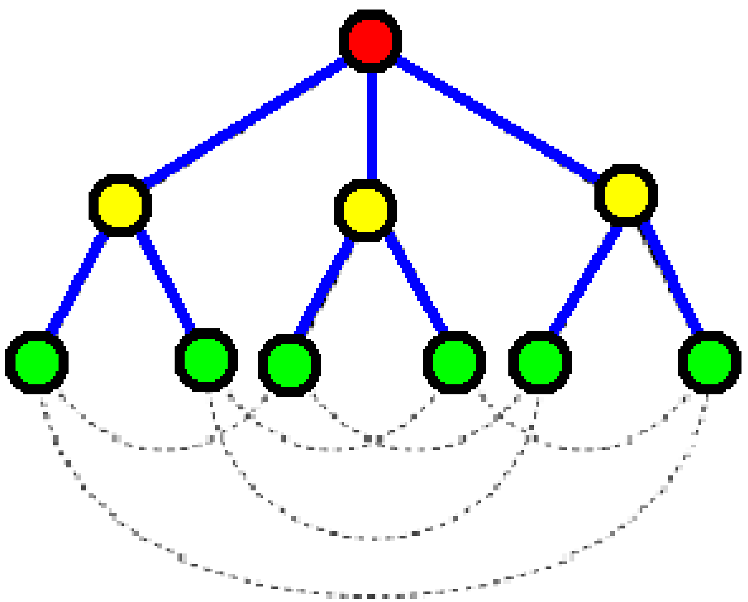


The Moore Bound:

(3,5)-cage: 10

g odd:

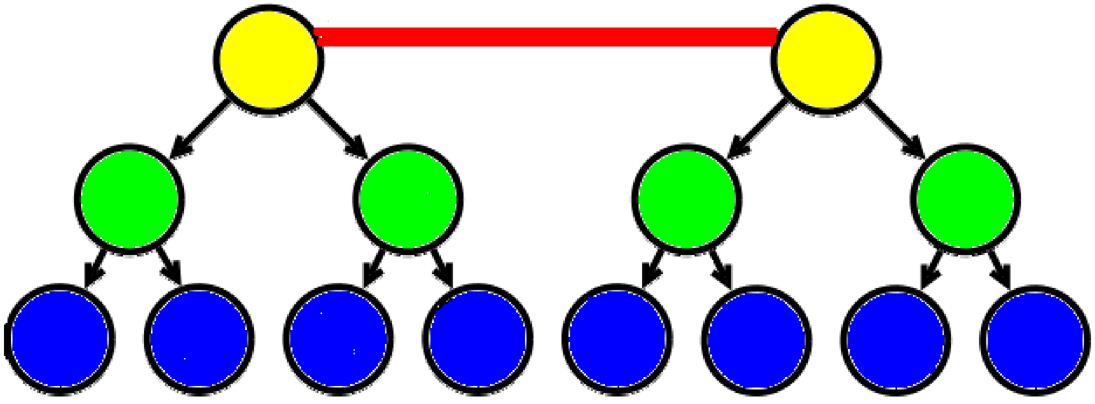
$3 * 2^{(g-1)/2} - 2$



(3,6)-cage: 14

g even:

$2^{(g+2)/2} - 2$



Moore graphs: cages which satisfy the Moore bound.

(d, 2k)-cages:

$$1 + d\sum_{i=0}^{k-1}(d-1)^i.$$

(d, 2k+1)-cages:

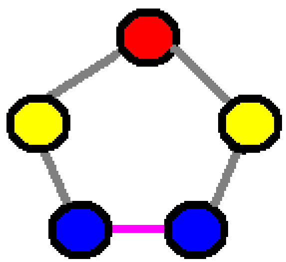
$$2\sum_{i=0}^{k-1}(d-1)^i = 1 + (d-1)^{k-1} + d\sum_{i=0}^{k-2}(d-1)^i.$$

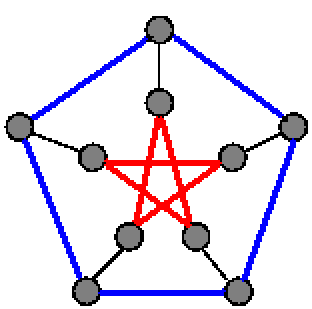
Hoffman-Singleton theorem: any Moore graph with girth 5 must have degree 2, 3, 7, or 57.

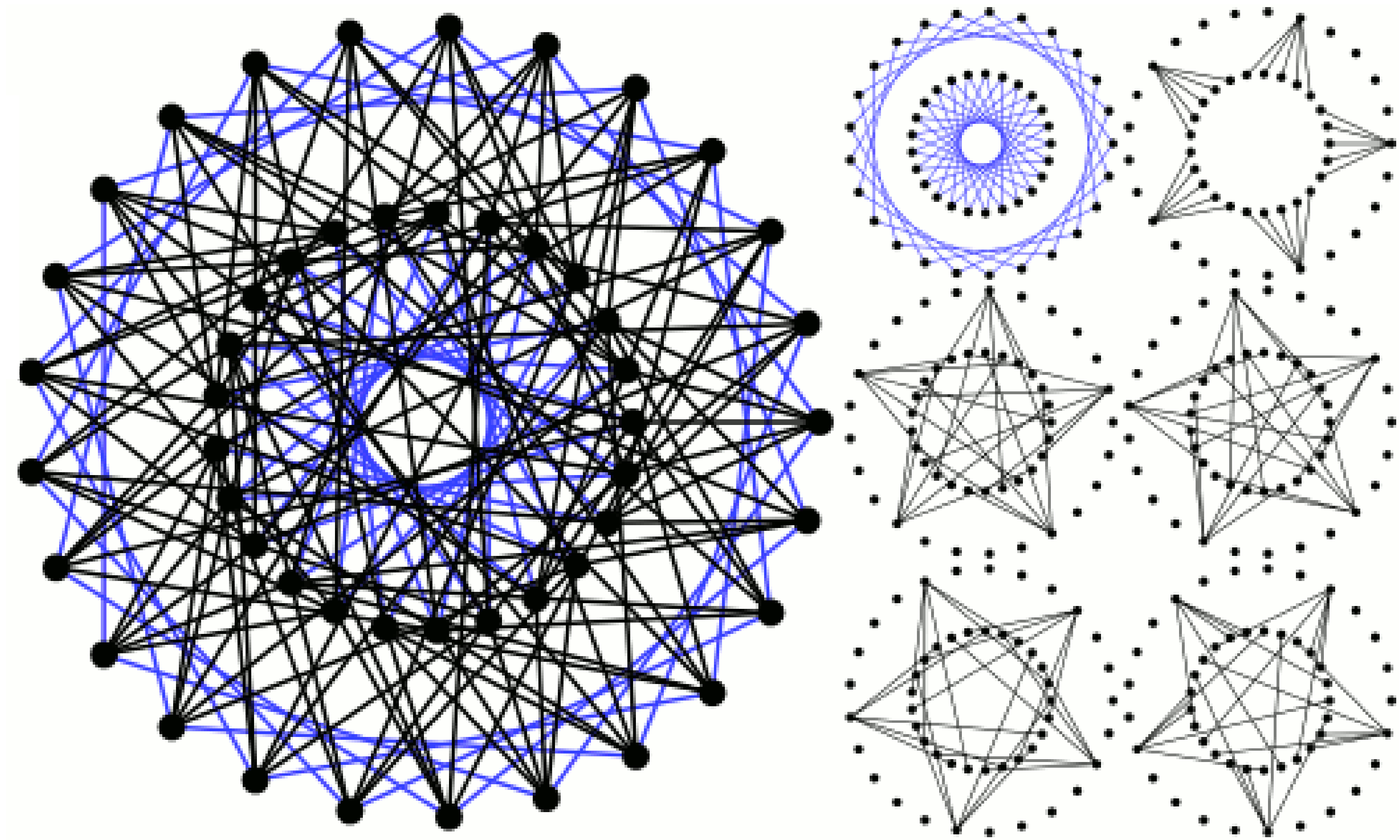
Proof: Uses eigenvalues of $B = A^2 + A$ where A is the adjacency matrix. The graphs with degree 2, 3, and 7 are the pentagon, Petersen graph, and Hoffman-Singleton graph, respectively.

BIG OPEN QUESTION: Does a Moore graph with girth 5 and degree 57 exist?

Moore graphs with girth 5 and degrees 2, 3, 7:







Hoffman-Singleton graph

There are 18 (3,9)-cages, and v(3,9) = 58. The first such cage was found by Biggs & Hoare (1980), the fact that v(3,9) = 58 and the remaining examples are due to Brinkmann, McKay & Saager (1995). Verified in new work (SODA).

There are 3 (3,10)-cages, all bipartite, and v(3,10) = 70. This is due to O'Keefe & Wong (1980).

http://www.win.tue.nl/~aeb/graphs/cages/cages.html

More recent work (in collaboration with Exoo):

(3,11)-graph of order 112 found by Balaban in 1973 is minimal and unique.

The order of a (4,7)-cage is 67 and we give one example.

Improved the lower bounds on the orders of (3,13)-cages and (3,14)-cages to 202 and 260, respectively.

Up to date cage info (Gordon Royle):

http://mapleta.maths.uwa.edu.au/~gordon/remote/cages/index.html

Combinatorial data:

http://mapleta.maths.uwa.edu.au/~gordon/

Small graphs, Small multigraphs
Cubic graphs
Symmetric cubic graphs (Foster Census)
Vertex-transitive graphs
Cayley graphs (by group)
Vertex-transitive cubic graphs
Cubic Cages and higher valency cages
Planar graphs

More info on cages:

http://school.maths.uwa.edu.au/~gordon/remote/cages/index.html

Cubic cages of small girth

The Cages	Smallest Known	n(3,g)	Number	Reference
(3,3)-cages	4	4	1	K_4
(3,4)-cages	6	6	1	K_3,3
(3,5)-cages	10	10	1	Petersen
(3,6)-cages	14	14	1	Heawood
(3,7)-cages	24	22	1	McGee graph
(3,8)-cages	30	30	1	Tutte's 8-cage
(3,9)-cages	58	46	18	Brinkmann/McKay/Saager
(3,10)-cages	70	62	3	O'Keefe/Wong
(3,11)-cages	112	94 [112]	1	McKay/Myrvold - Balaban
(3,12)-cages	126	126	1	Generalized hexagon

Cubic cages of small girth

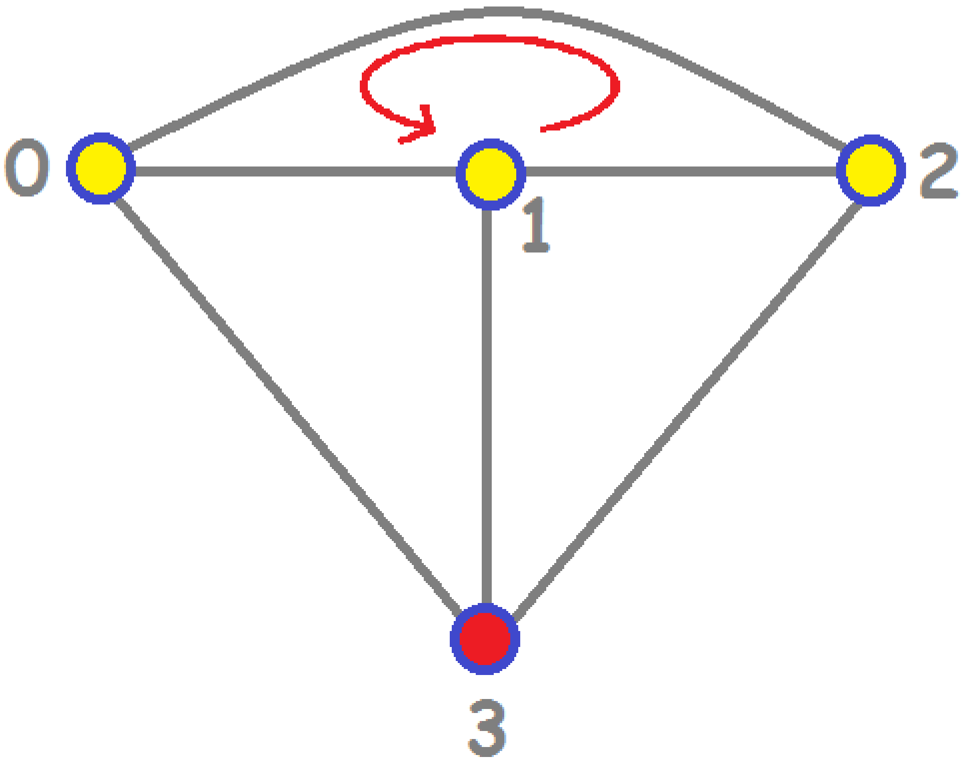
The Cages	Smallest Known	n(3,g)	Number	Reference
(3,13)-cages	272	190 [202]	1+	McKay/Myrvold - Hoare
(3,14)-cages	384	254 [258]	1+	McKay - Exoo
(3,15)-cages	620	382	1+	Biggs
(3,16)-cages	960	510	1+	Exoo
(3,17)-cages	2176	766	1+	Exoo
(3,18)-cages	2640	1022	1+	Exoo
(3,19)-cages	4324	1534	1+	H(47)
(3,20)-cages	6048	2046	1+	Exoo

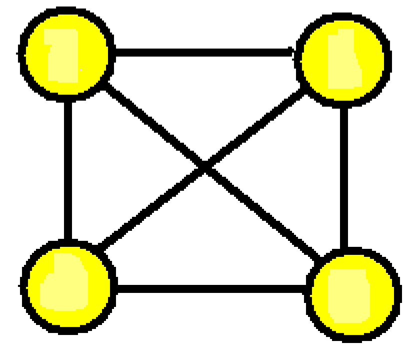
There is a BIG gap between the best available lower bounds on the number of vertices of a cage and the smallest graphs found so far with a given girth.

Auspicious target:
Search for a small 3-regular graph of girth 14 which has some symmetry.

Girth 13: 190-272
Girth 14: 258-384

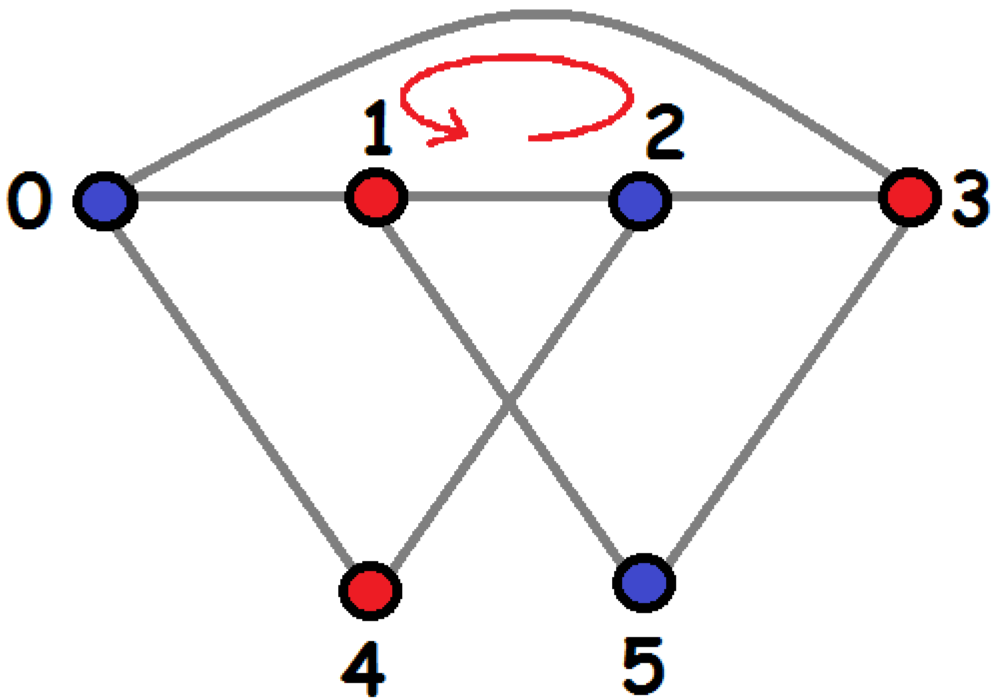
Redrawing some smaller cages to show one cyclic symmetry:

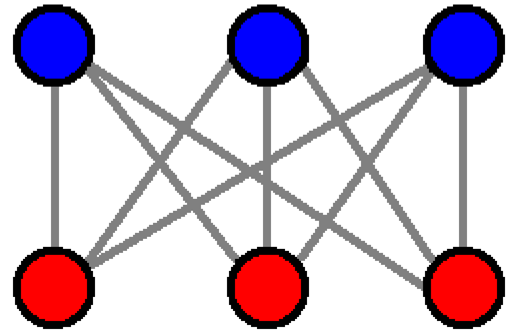




(012) (3)

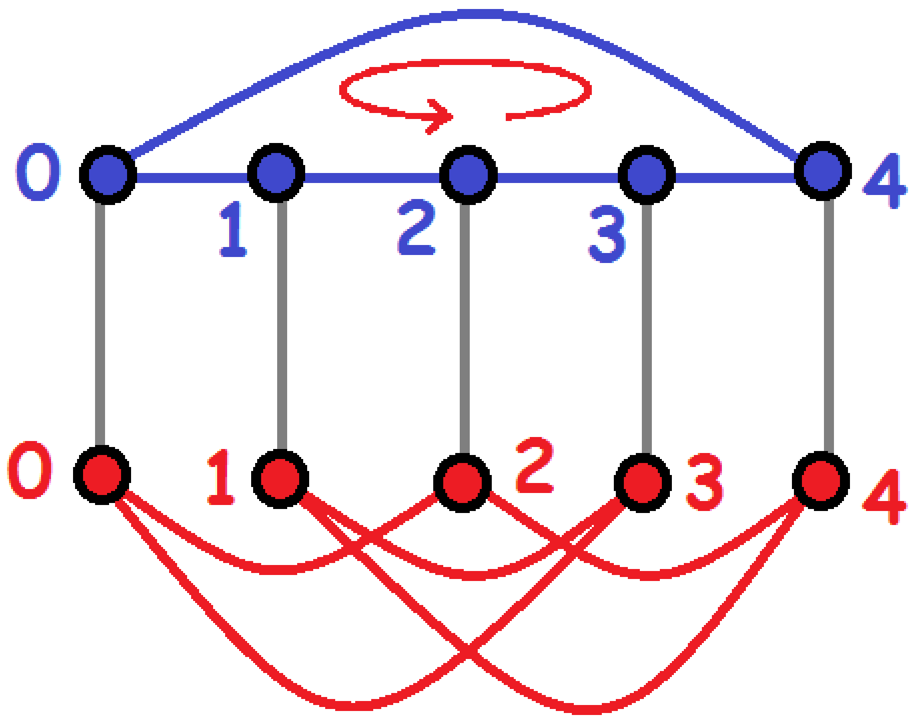
Redrawing some smaller cages to show one cyclic symmetry:

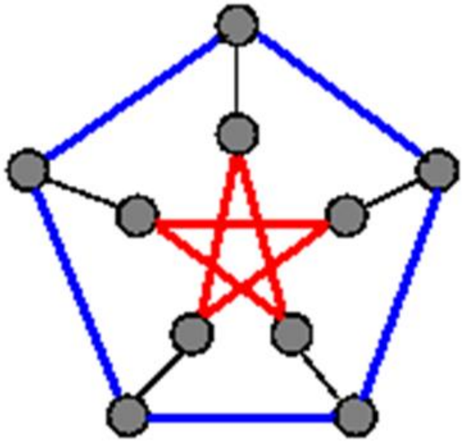




(0123) (45)

Redrawing some smaller cages to show one cyclic symmetry:

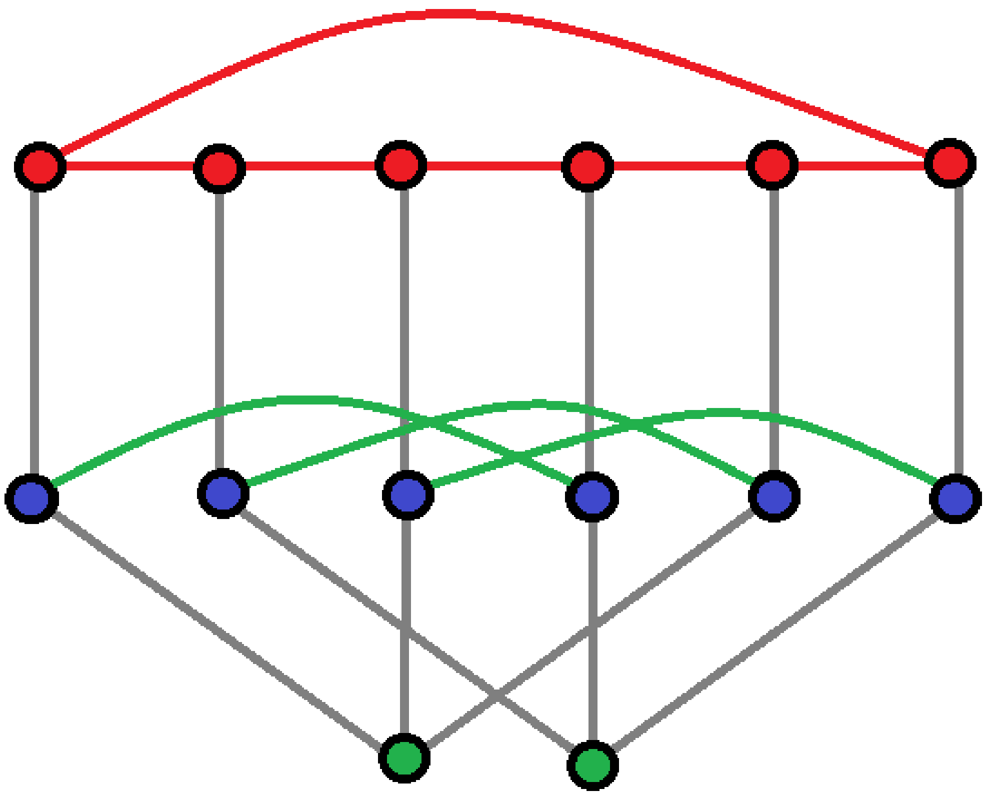


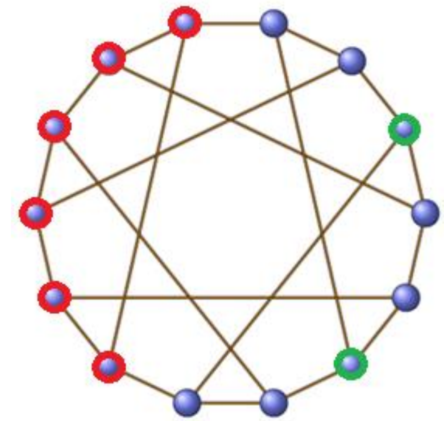


Edges from
i to i+2
(jump 2)

(01234) (01234)

Redrawing some smaller cages to show one cyclic symmetry:



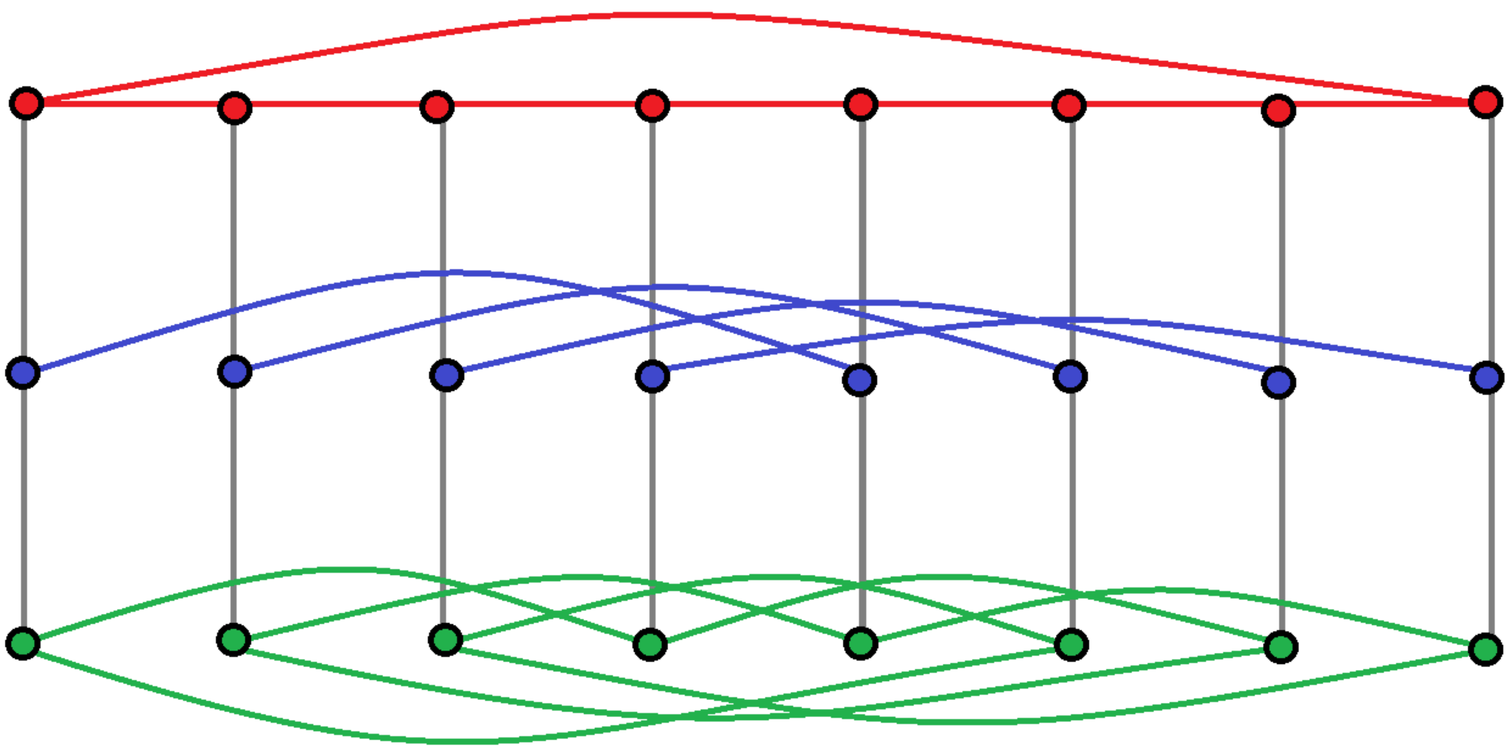


matching edges jump 3

(012345) (012345) (01)

McGee Graph: Note that the girth is 7 but there is an 8-cycle at the top level. Obvious lower bound=22, n= 24 for (3,7)-cage.

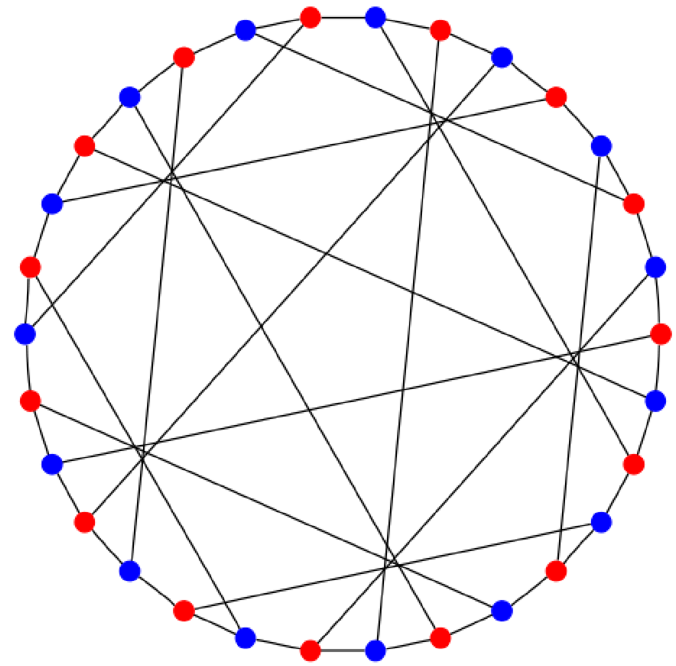
Matching jump 4, cycle jump 3:



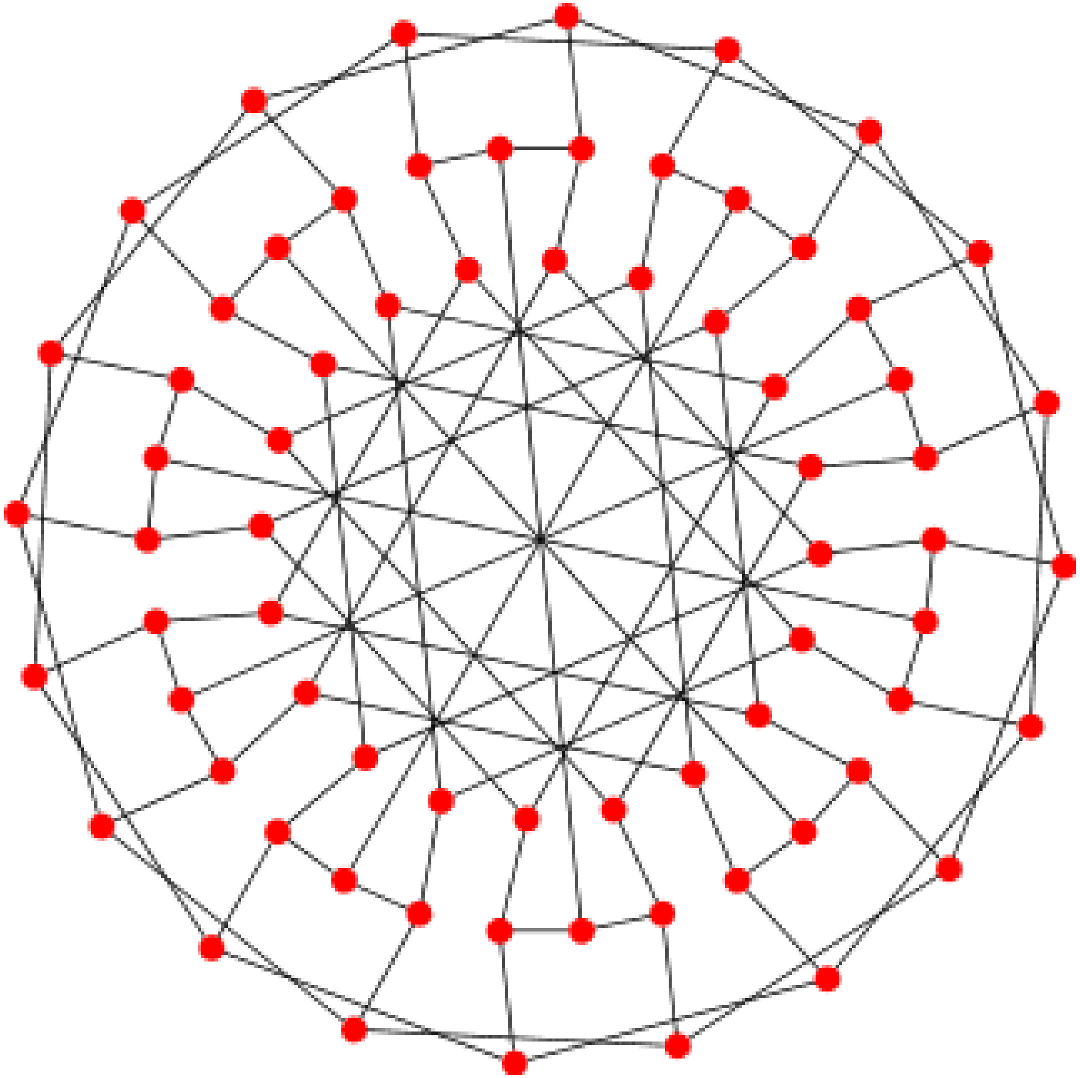
10-cycle: one down edge per vertex.

Matching with jump=5, one down edge and one up edge per vertex.

Cycle with jump 3: one up edge per vertex.



Tutte-Coxeter graph (3,8)-cage.



Balaban's 10-cage from wikipedia.

Computation time

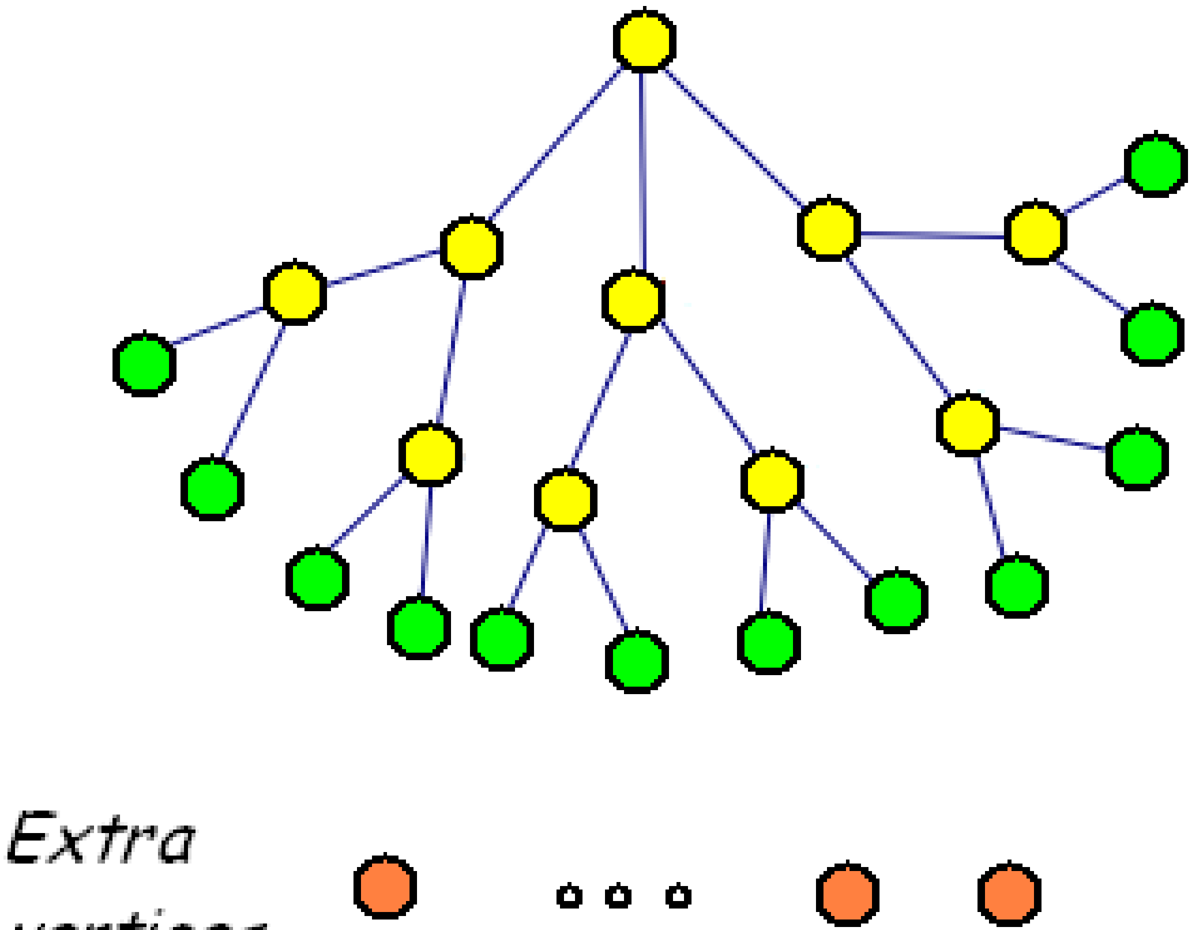
Girth	Result	maus	new
9	18 cages	259 days	5 days
11	104 bad	2.5 years	25.6 hours
	112		6.7 years
13	> 200		2.3 years
14	> 256		18.8 hours

Same environment since McKay involved in both projects.

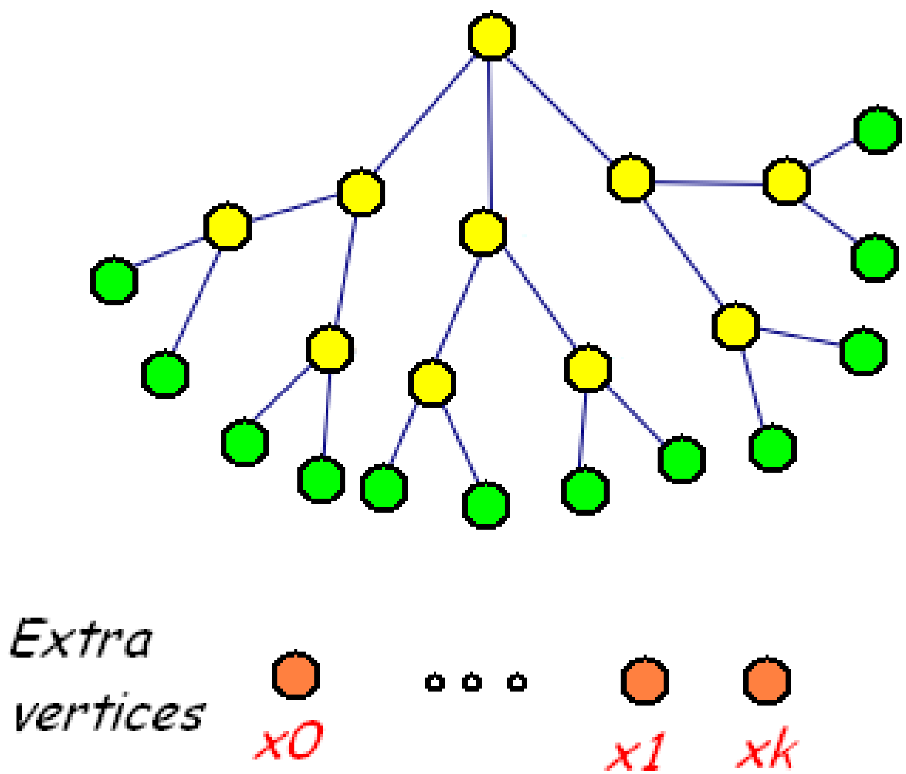
Backtracking Rules of Thumb

1. Start with what you know.
2. Do strong redundancy checks near the root of the search tree, but only fast checks in other places.
3. If there are choices to be made, select a decision with a minimum number of options.
4. Abort early if possible.
5. Do as little as possible at each recursive call.
6. Keep it simple if you can.
7. Distribute work by sending branches of the computation tree to various machines.

1. Start with what you know.



2. Do strong redundancy checks near the root of the search tree, but only fast checks in other places.



Fast check:

If x0, x1, ... xk are isolated vertices then

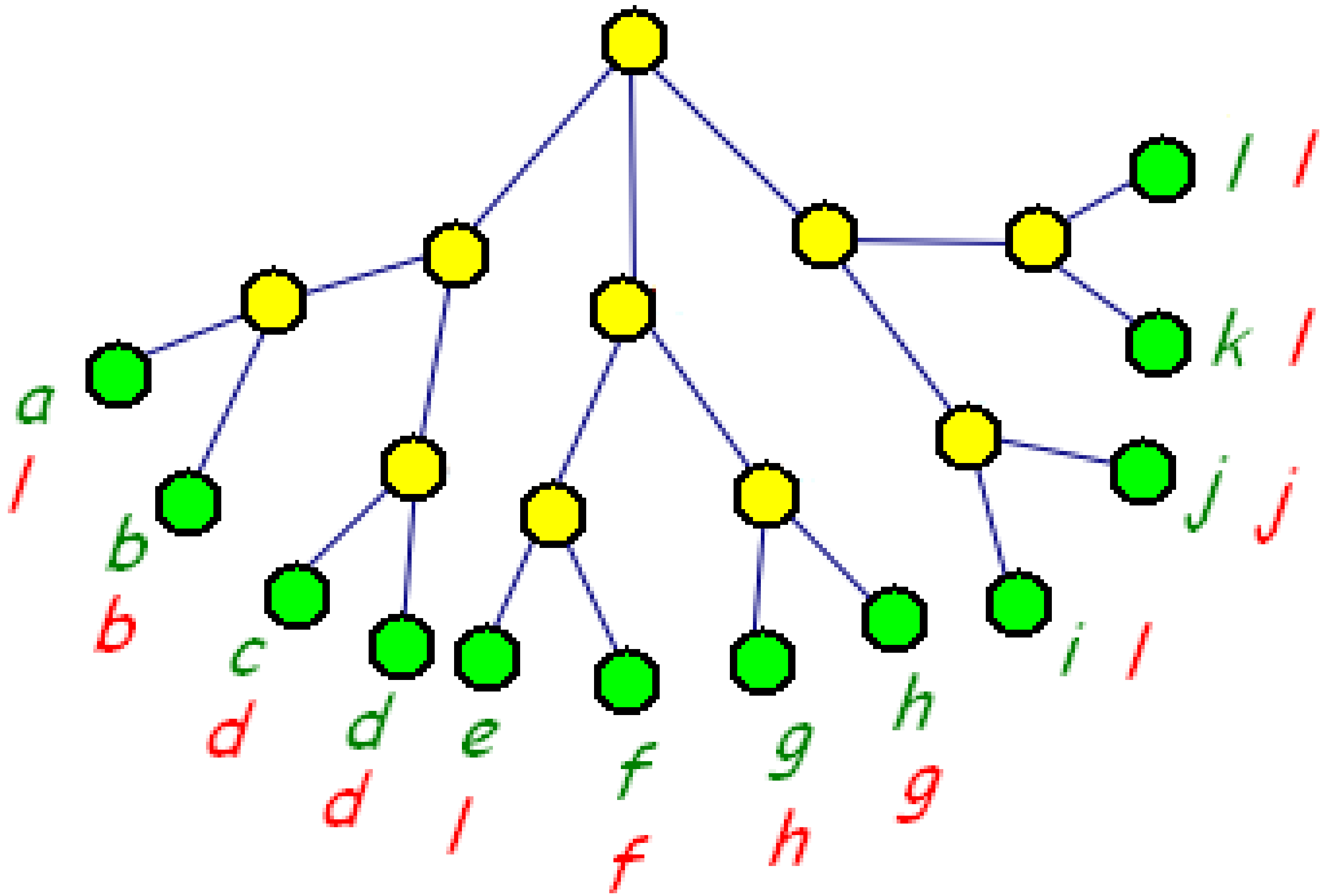
G + (v, x0) =

G + (v, x1) =

...

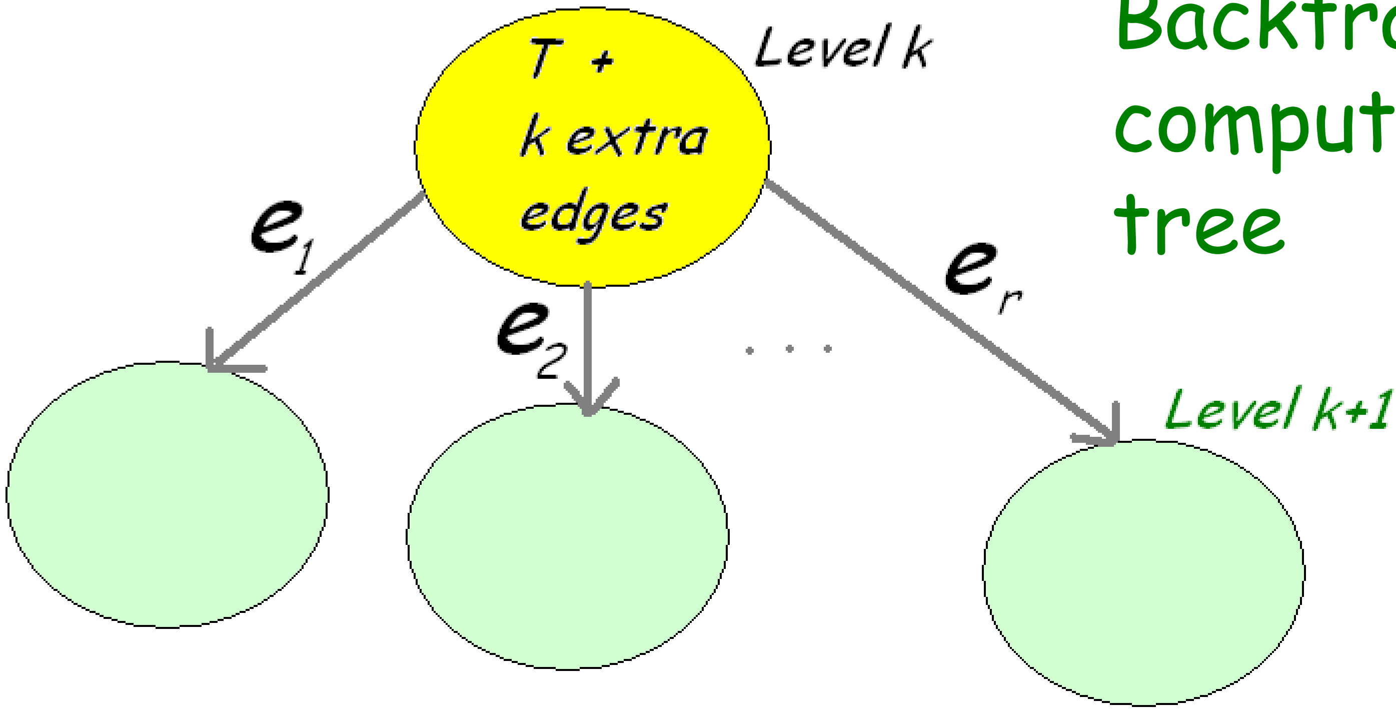
G+(v, xk).

Make leftmost choice of isomorphic alternatives.



Extensive redundancy checks:

Label nodes: c_1, c_2, … c_k where c_i is choice made in going from level i-1 to level i. Maintain: each node enumerates all cages which contain the current graph as a subgraph not enumerated by some lex. earlier portion of the tree.



Backtrack computation tree

3. If there are choices to be made, select a decision with a minimum number of options.

Decisions: add an edge incident to some vertex v which has degree(v) < 3.

Edges which are legal to add are recorded.

We determine a decision with a minimum number of options. (not done in maus).

4. Abort early if possible.

If some vertex v with degree(v) < 3 has no choices for an incident edge, back up.

Note: there may still be some way to add edges preserving girth.

6. Keep it simple if you can.

Data structure:

Distance matrix indexed by vertices v such that degree(v) < 3. Decreases in size as you move away from the root of the backtrack computation.

Distance algebra:

1. Add (u, v) to G:

$$d(x,y) = \text{Min}\{d(x, y),\ d(x, u) + 1 + d(v, y),\ d(y, u) + 1 + d(v, x)\}$$

2. Values $\geq g\text{-}1 = \infty = g\text{-}1$.

3. BAD= $g\text{-}2$. Used for values $\geq g\text{-}2$ where an edge should not be added due to isomorphism constraints.

7. Distribute work by sending branches of the computation tree to various machines.

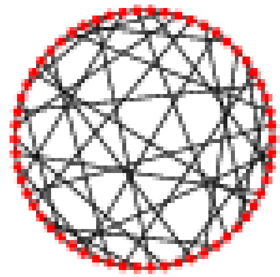
Cut computation tree at a particular level.

Label subproblems:

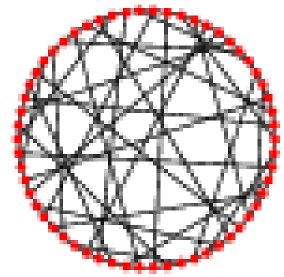
0..M 0..M 0..M 0..M … 0..M
Not
0000 11111 … MMMM

Autoson was used to automatically distribute computation to various machines.

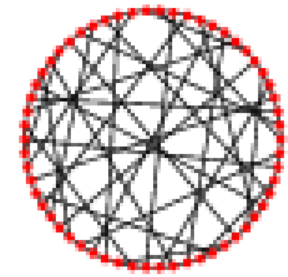
(3,9)-cage 1



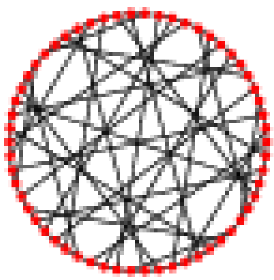
(3,9)-cage 2



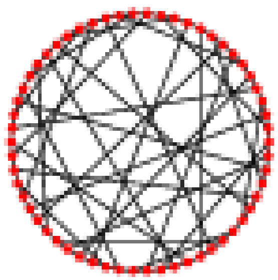
(3,9)-cage 3



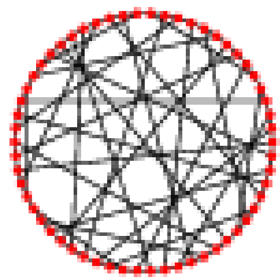
(3,9)-cage 4



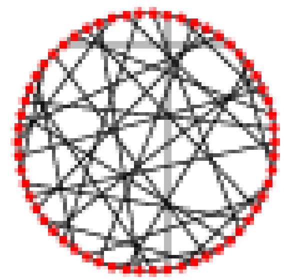
(3,9)-cage 5



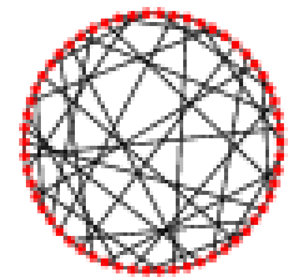
(3,9)-cage 6



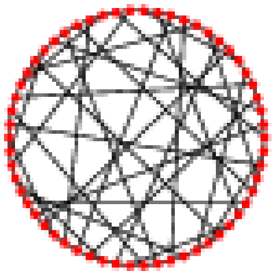
(3,9)-cage 7



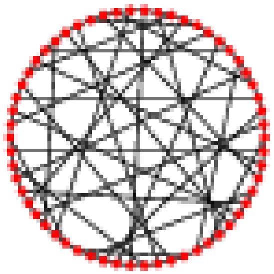
(3,9)-cage 8



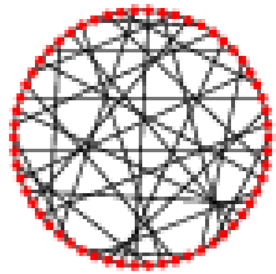
(3,9)-cage 9



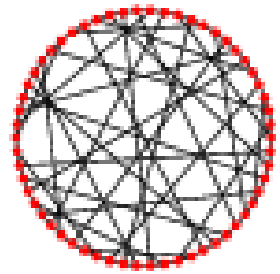
(3,9)-cage 10



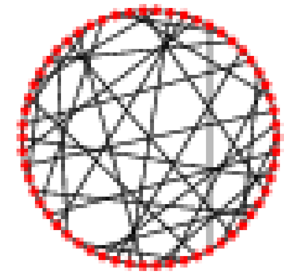
(3,9)-cage 11



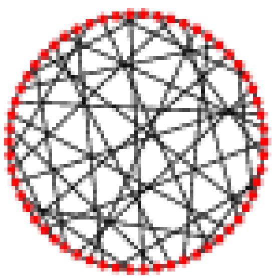
(3,9)-cage 12



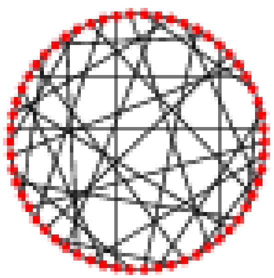
(3,9)-cage 13



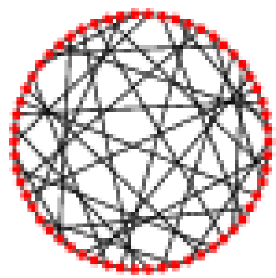
(3,9)-cage 14



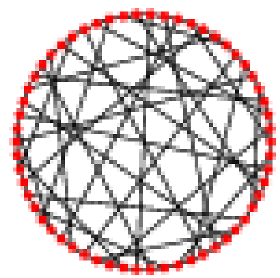
(3,9)-cage 15



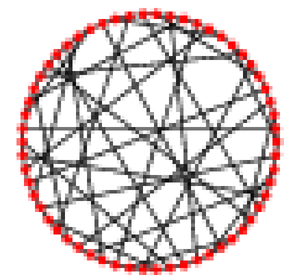
(3,9)-cage 16



(3,9)-cage 17



(3,9)-cage 18



Pictures from:

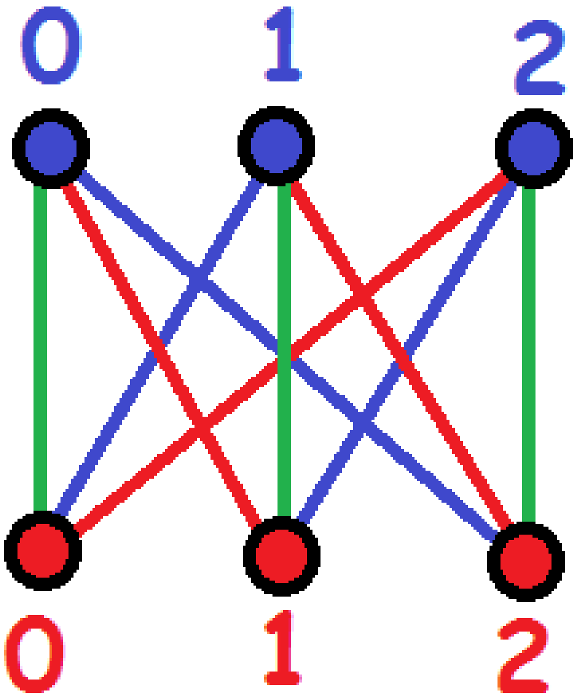
http://mathworld.wolfram.com/CageGraph.html

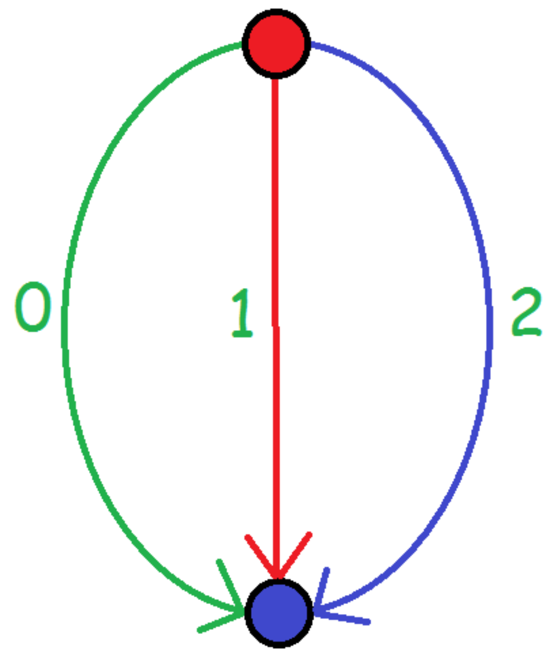
To search for graphs beating the current ones with symmetry:

Initial target: assume that there is some symmetry in the automorhism group that consists of all p cycles for some integer p.

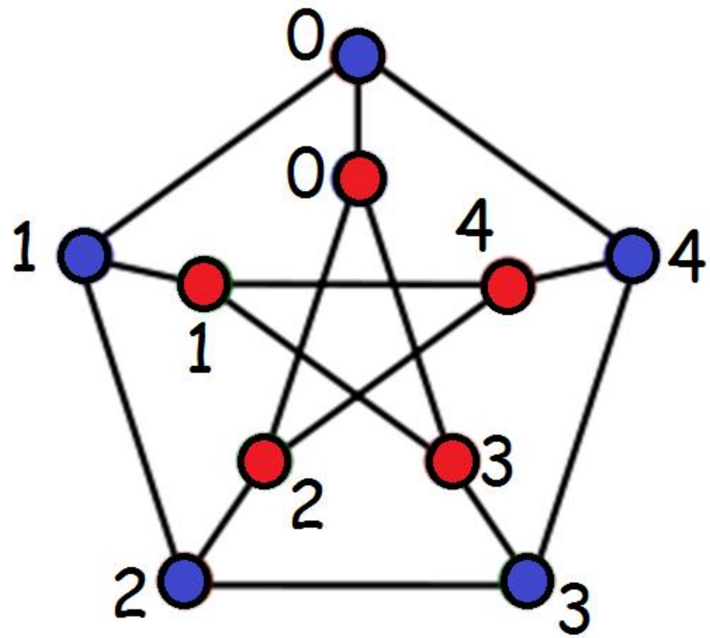
Step 1: re-express the known examples this way if we can.

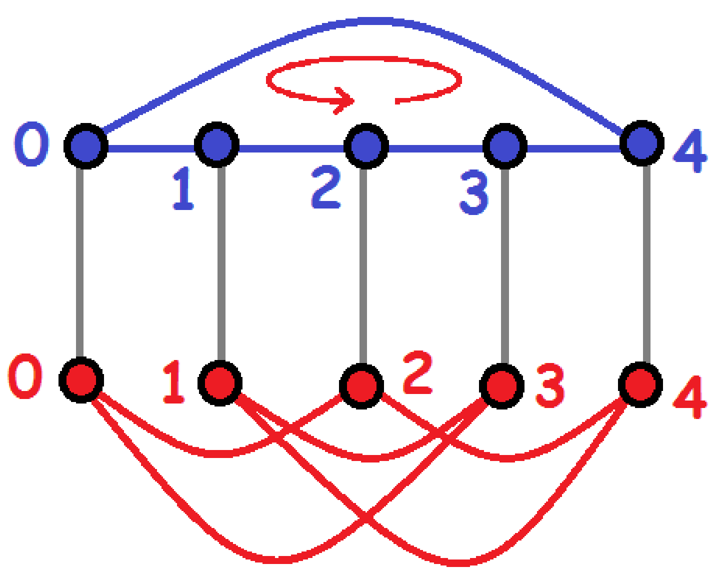
The (3,4)-cage: $K_{3,3}$

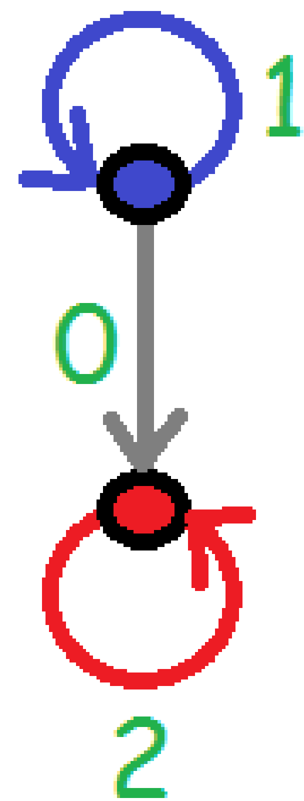




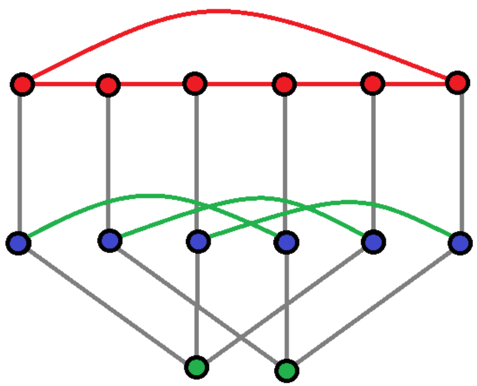
The Petersen graph
(3,5)-cage:

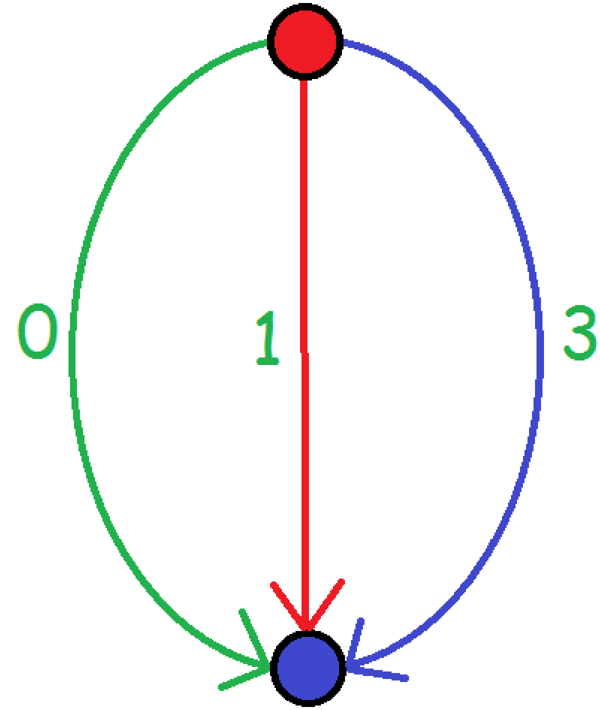


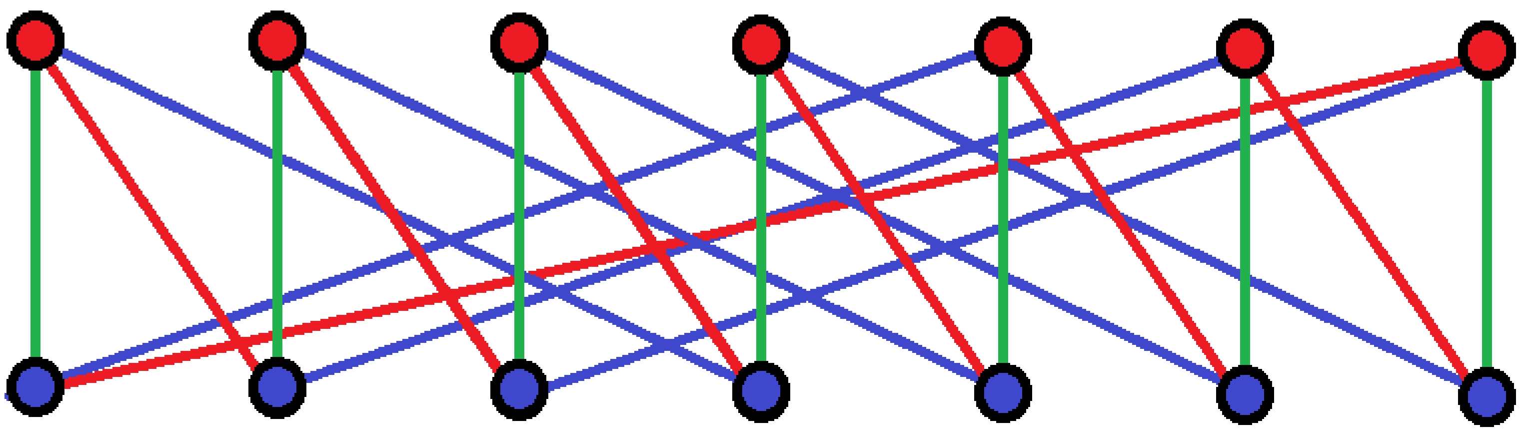




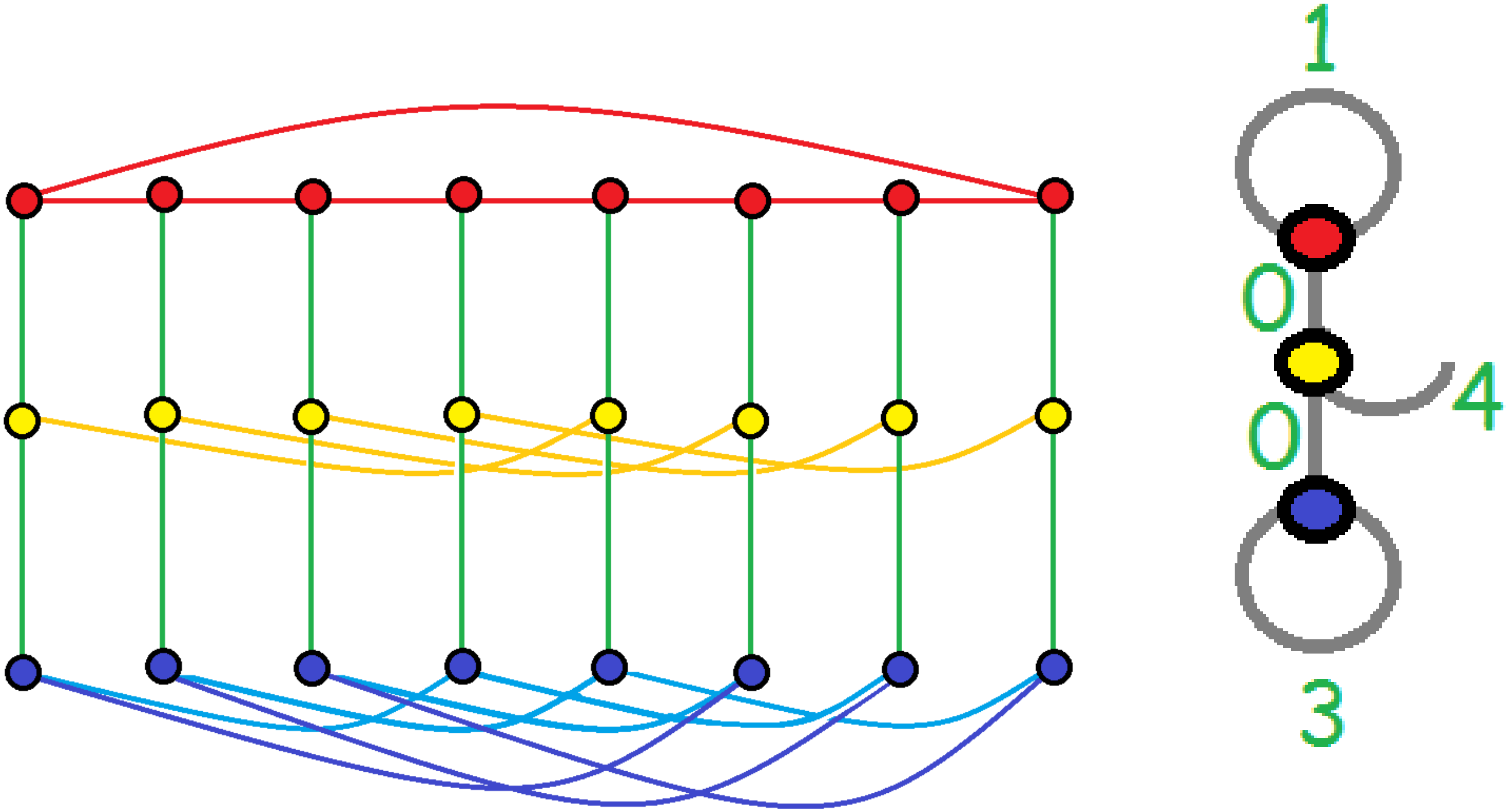
The (3,6)-cage:





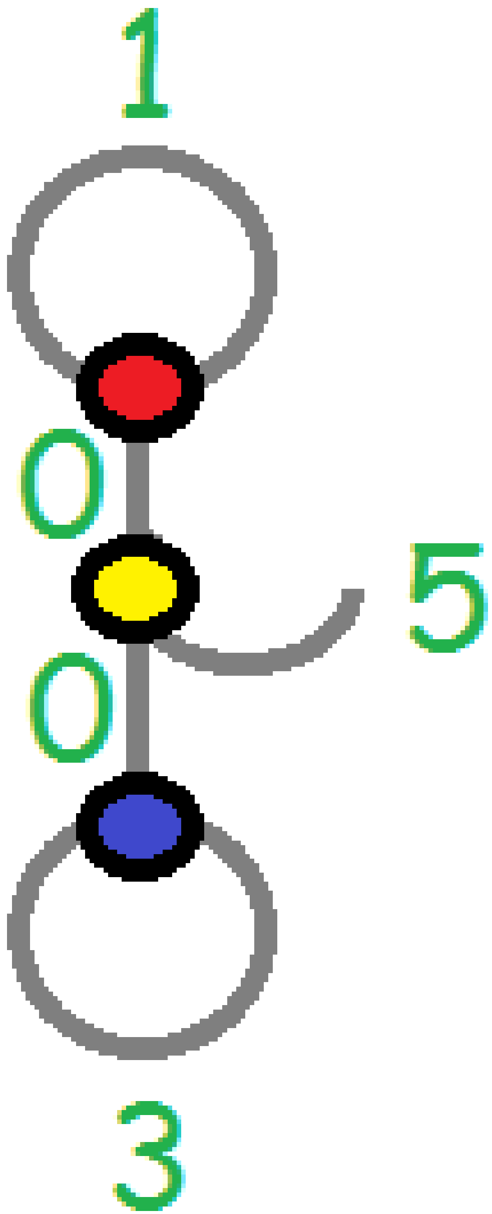


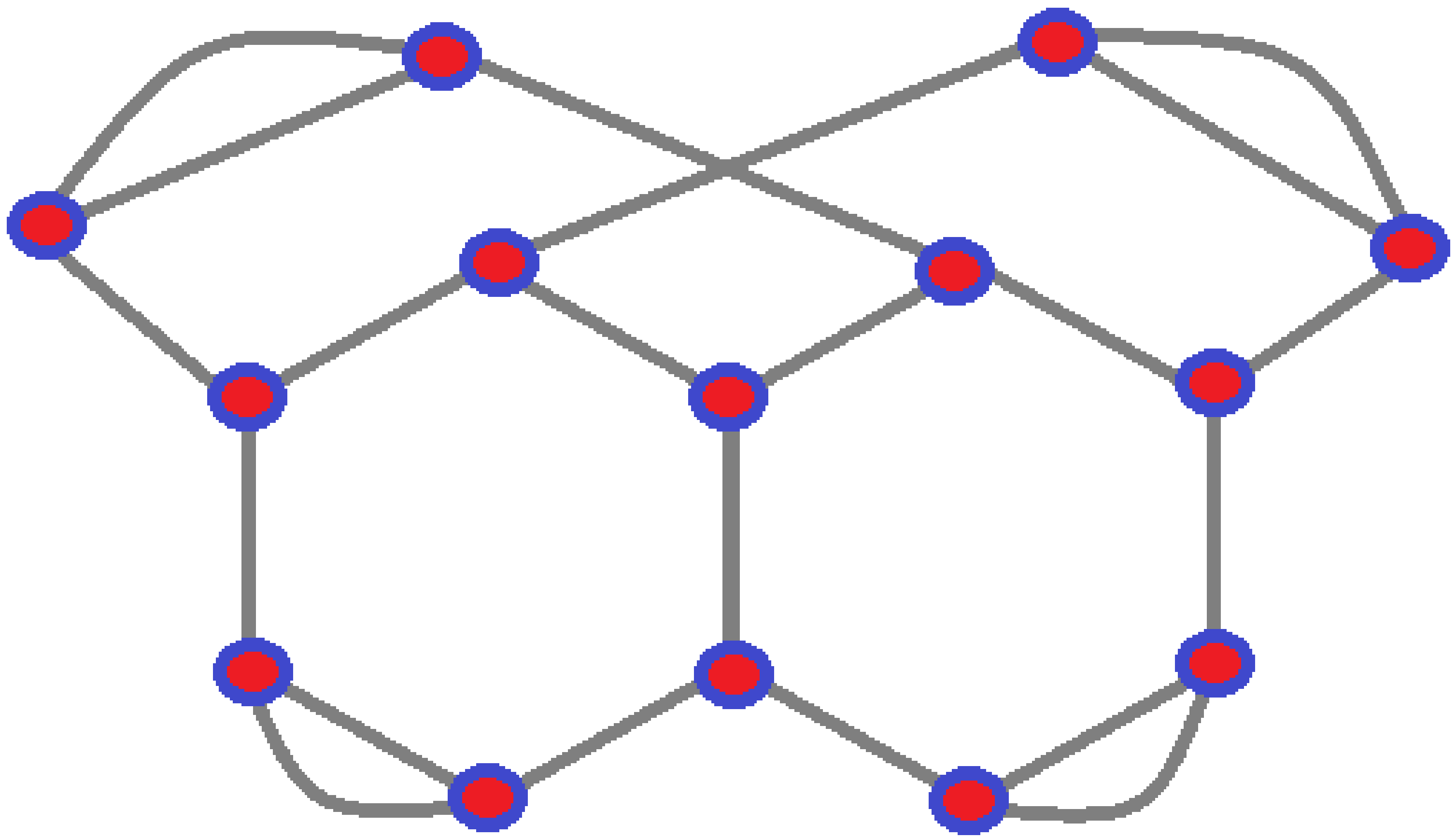
(3,7)-cage: 24= 3 * 8



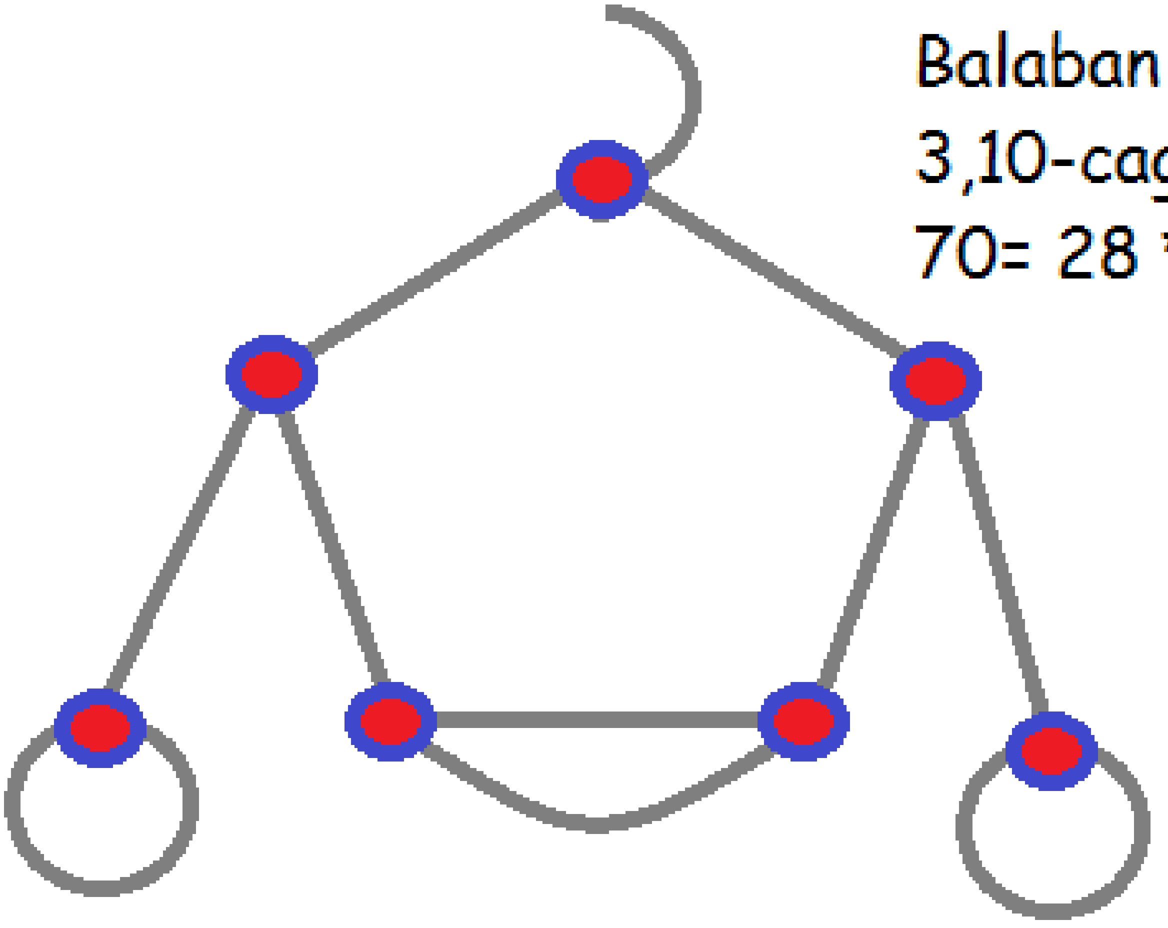
(3,8)-cage:

30= 3 * 10





Harries 3,10-cage: 70 = 14 * 5



Balaban
3,10-cage
70= 28 * 5

Cycle structures of the (3,10)-Wong cage are not amenable to this approach.

The automorphism group order: 24

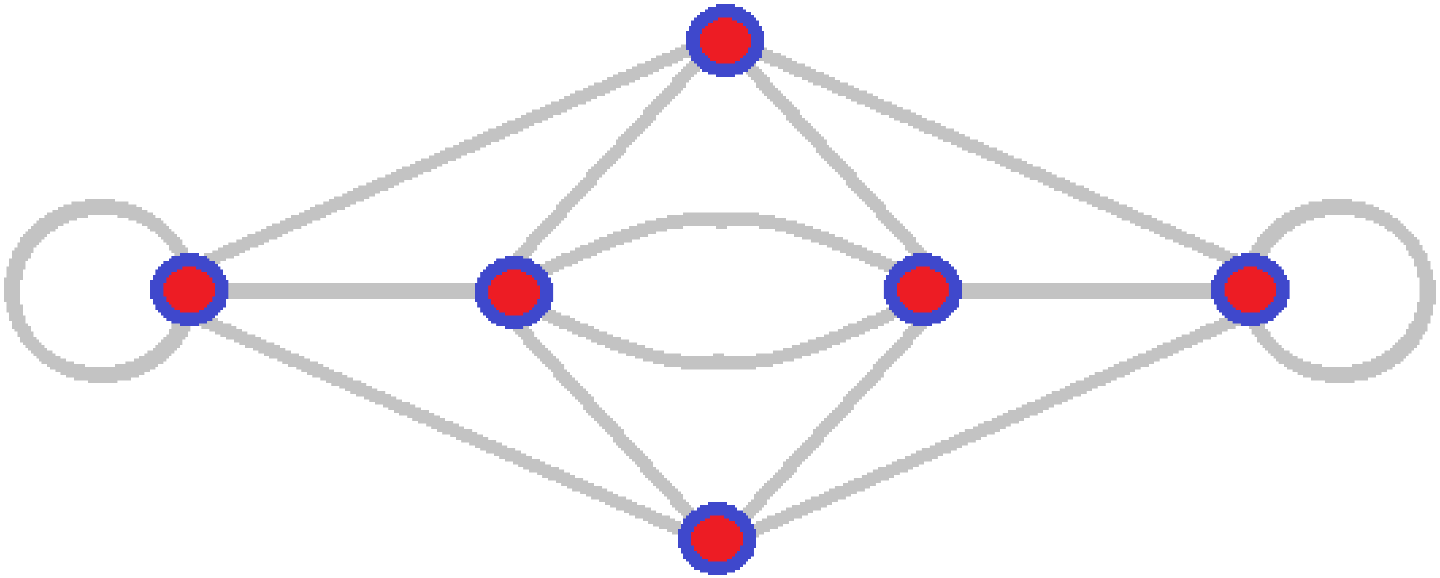
The types of permutations:

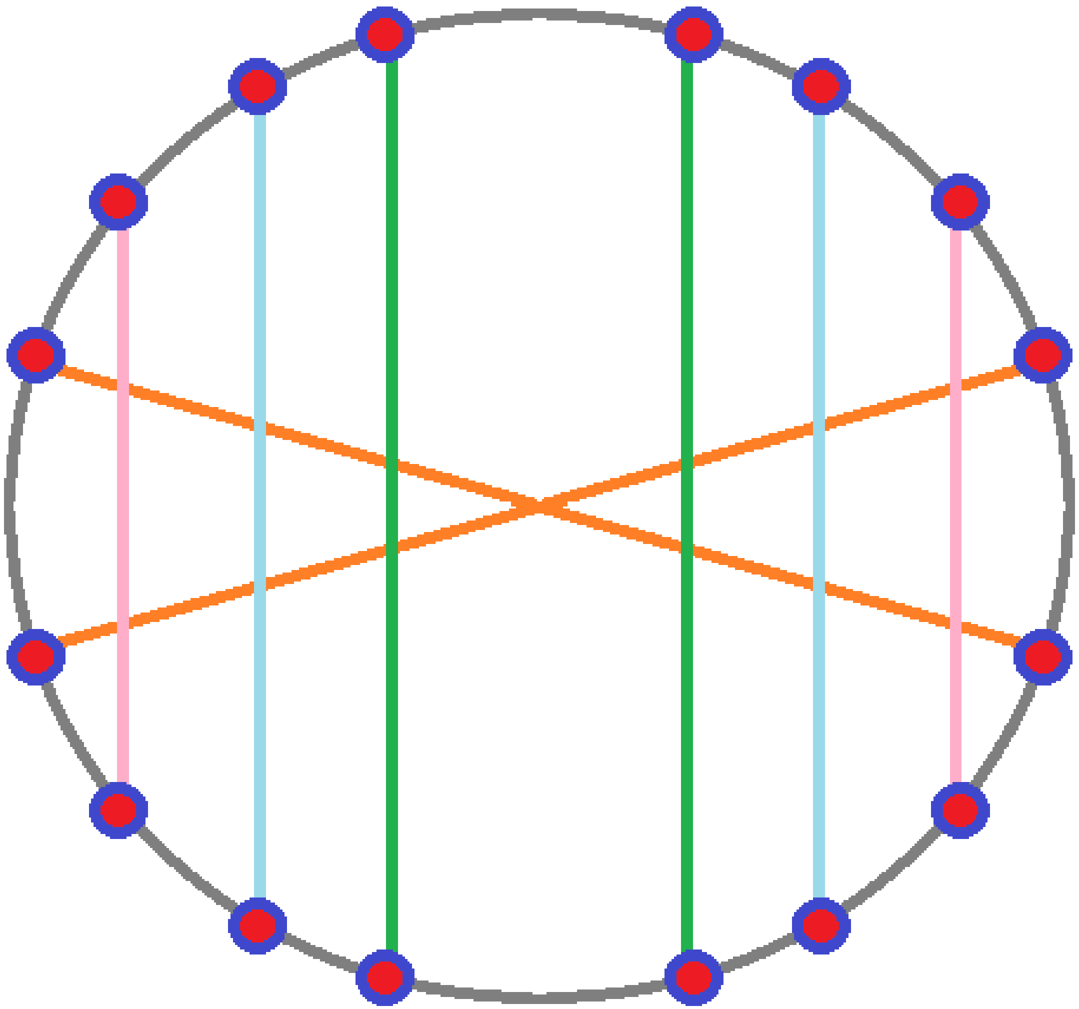
1. [1: 14] [2: 28]
2. [1: 2] [2: 6] [4: 14]
3. [1: 4] [3: 22] // 4 fixed points and 22 3-cycles
4. [1: 6] [2: 32]
5. [1: 70] // identity permutation.

Meaning of notation:

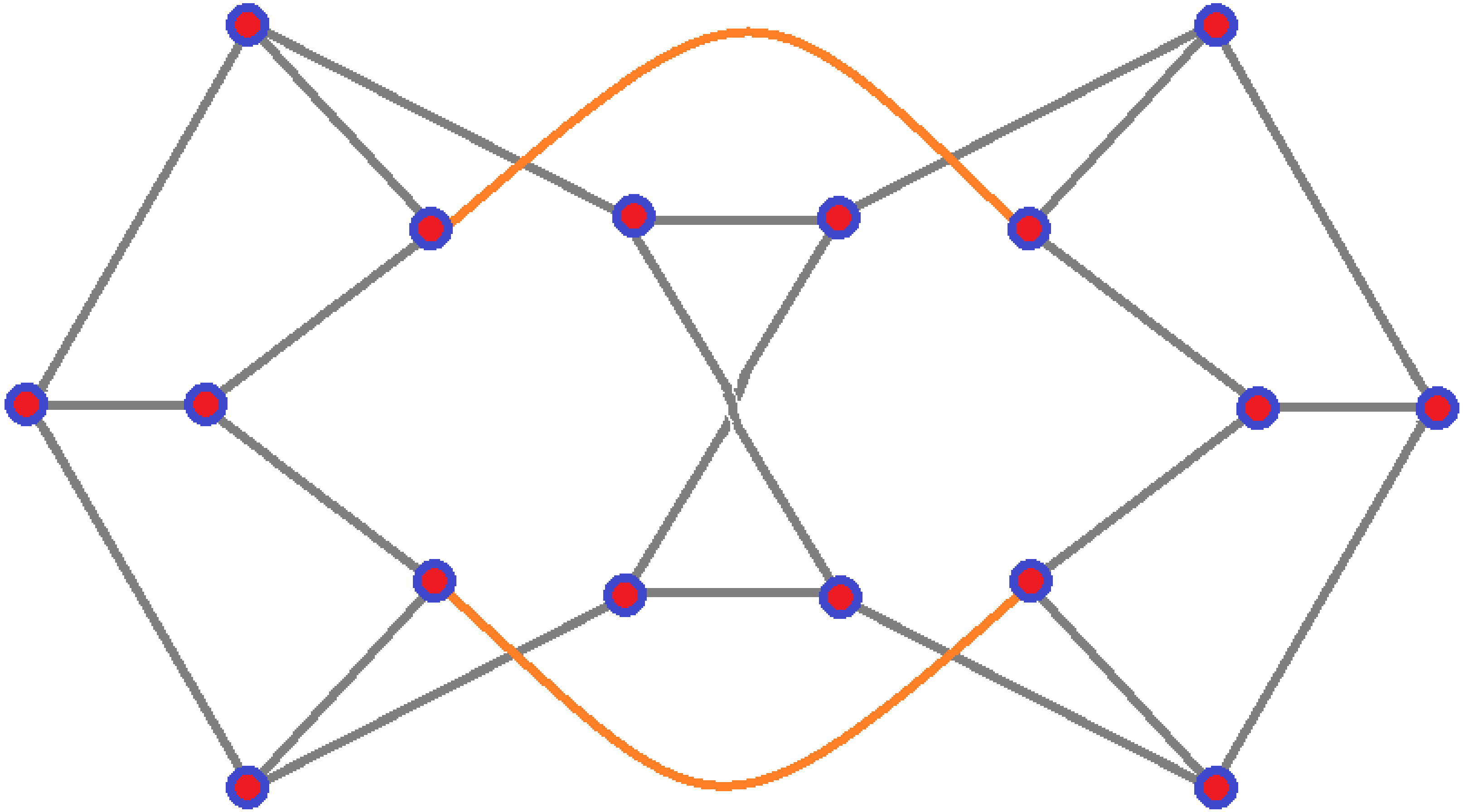
[cycle size: number of that cycle size]

(5,5)-cage: 30= 5*6

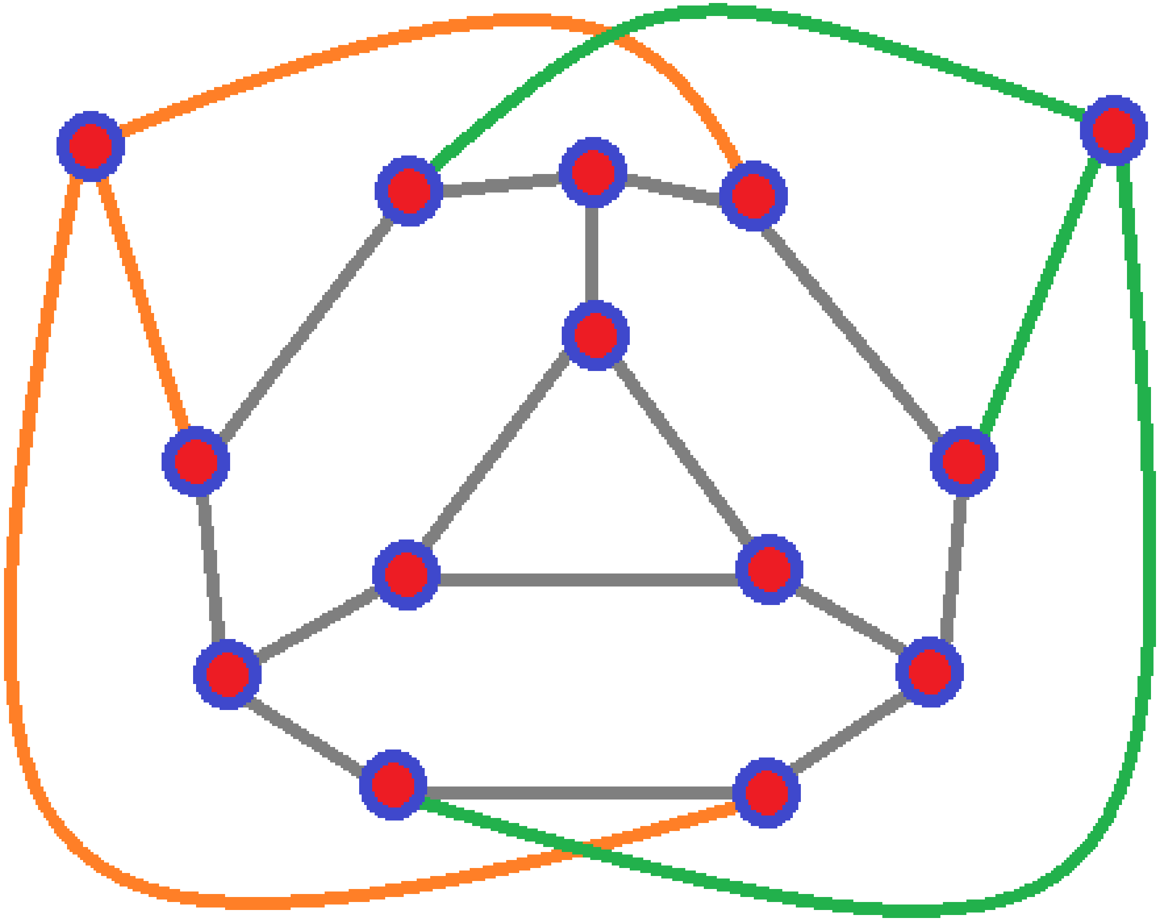




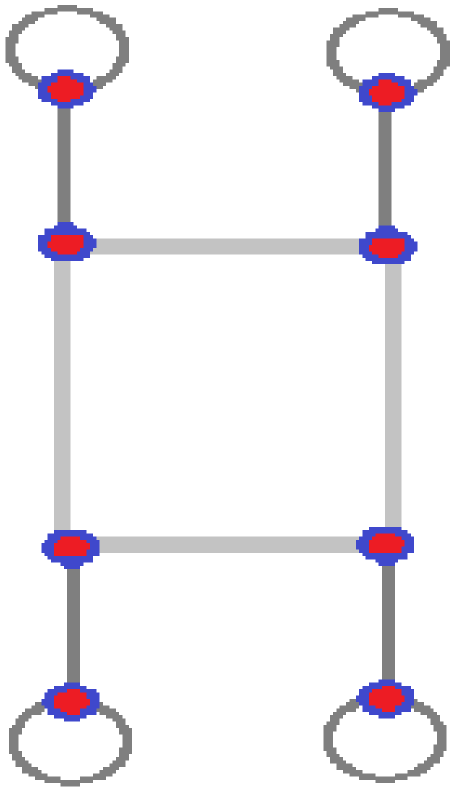
Current-best 3,13-graph: 272= 16 * 17



Current-best 3,14-graph: 384 = 16 * 24



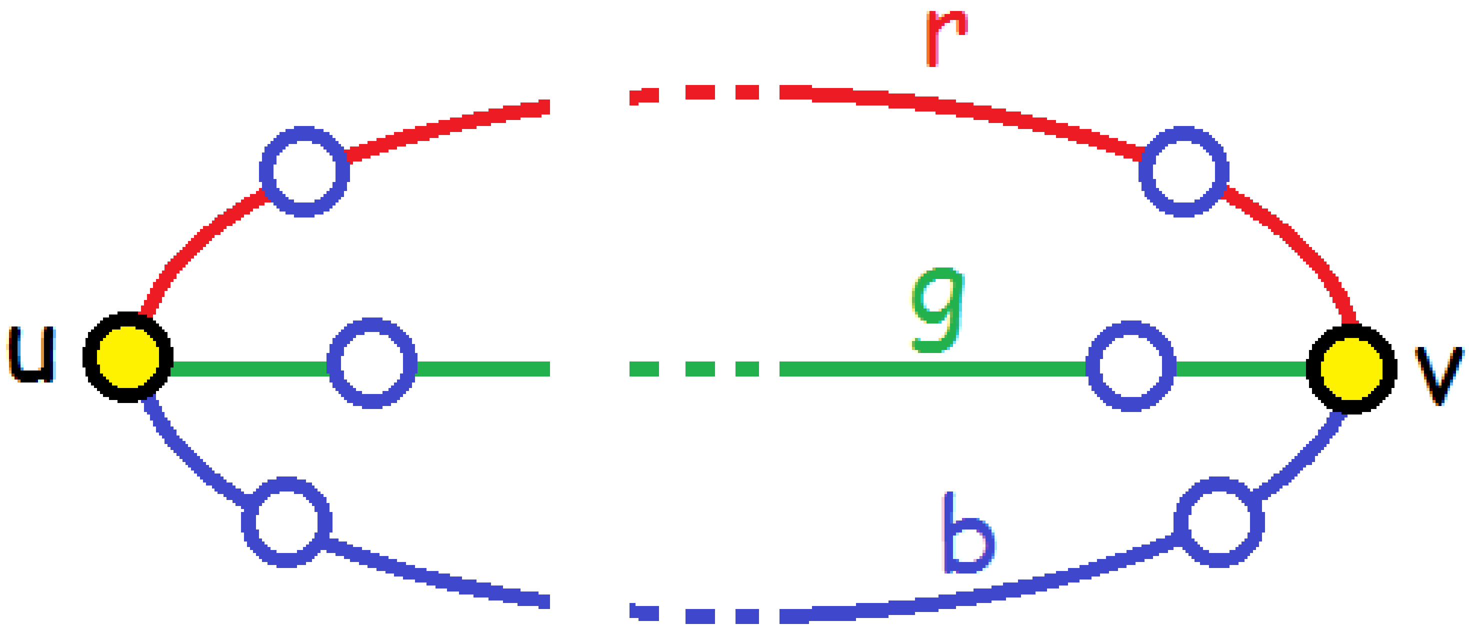
3,13-graph with 350= 14 * 35 vertices



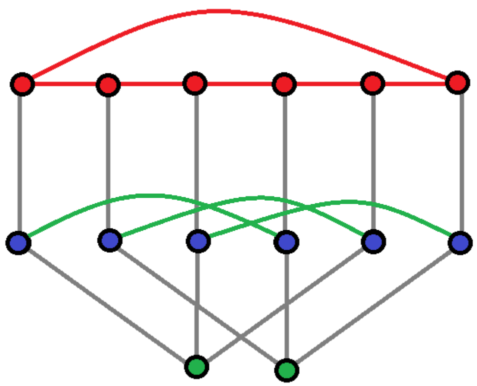
376= 8 * 47 vertices, girth 13

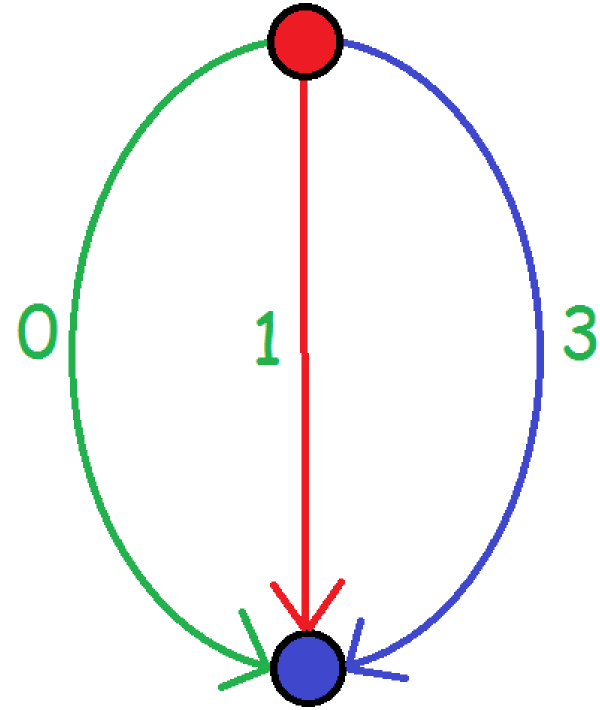
Cycles are 47-cycles.

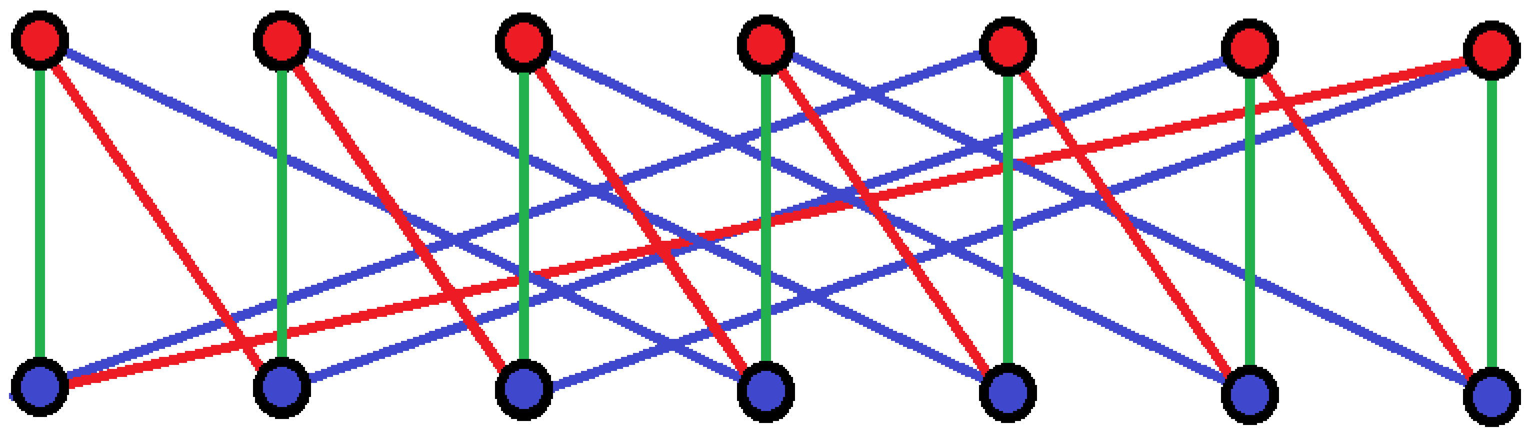
Bad theta graphs:



The (3,6)-cage:

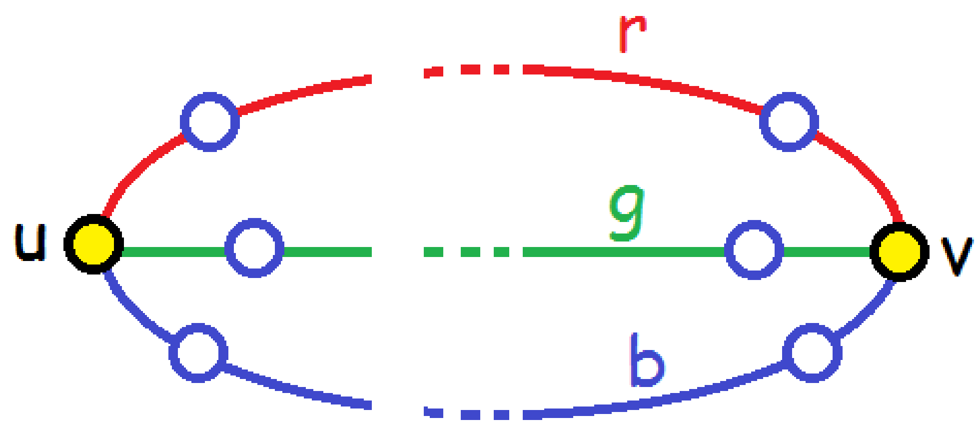


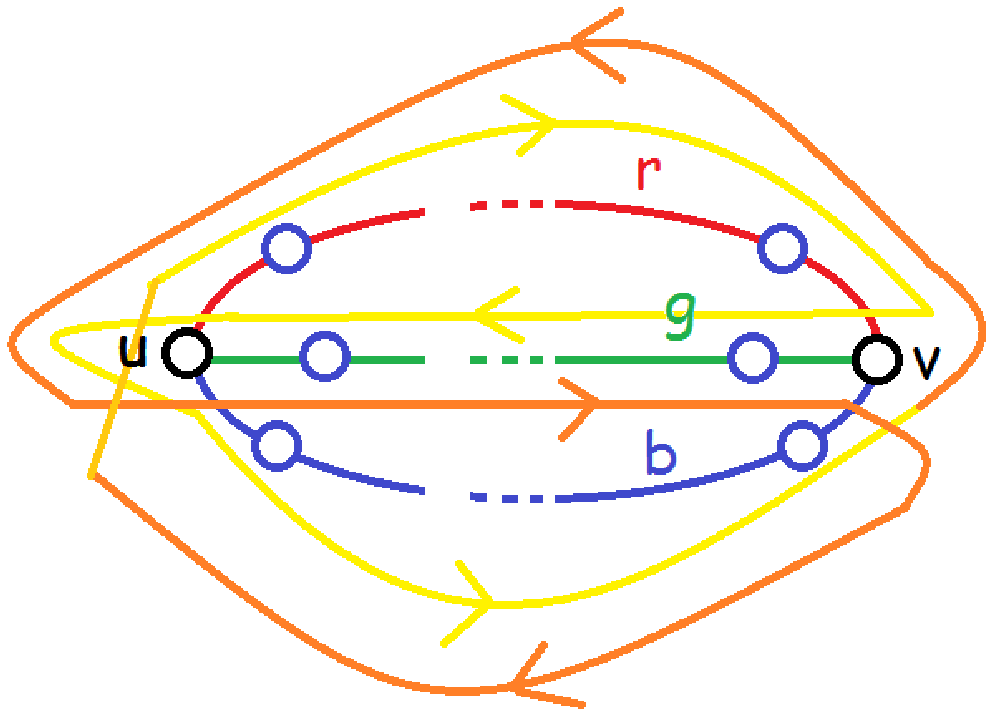


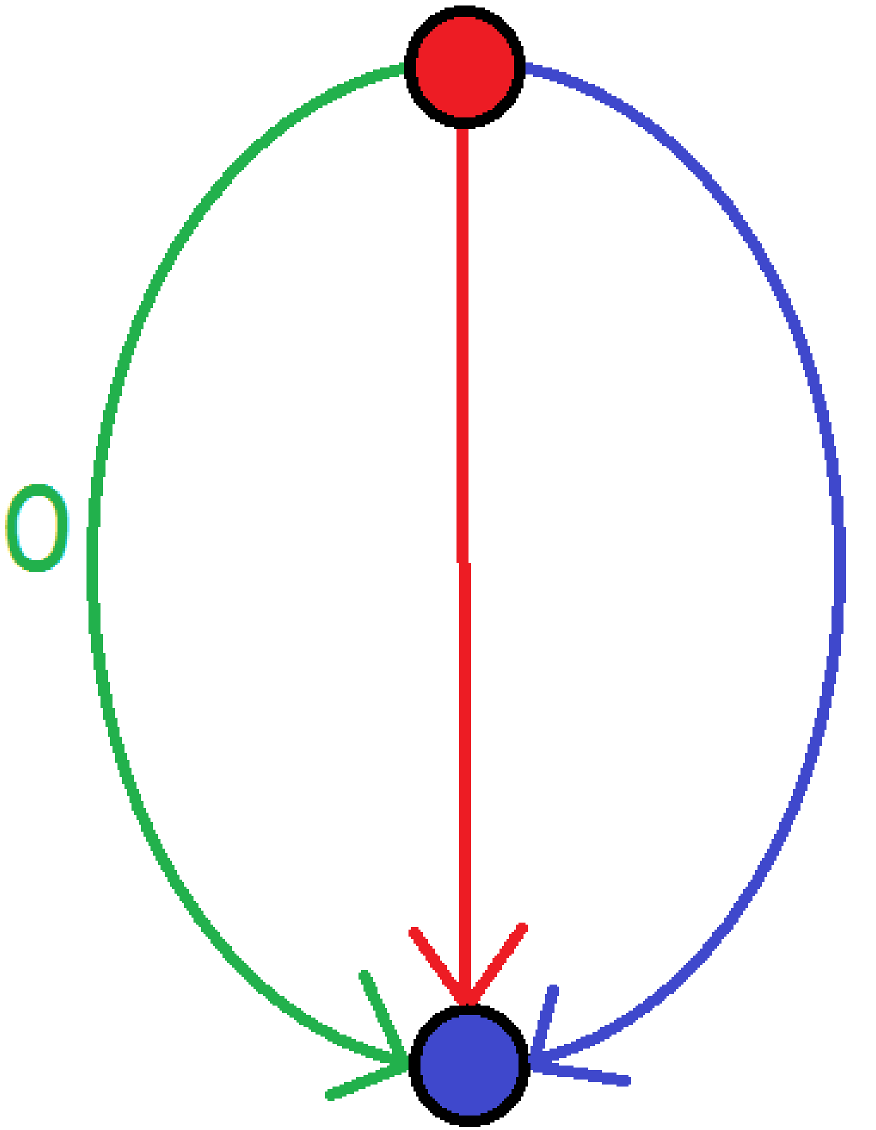


Bad theta subgraphs:

Girth is at most 2(r+g+b) no matter what jumps are used



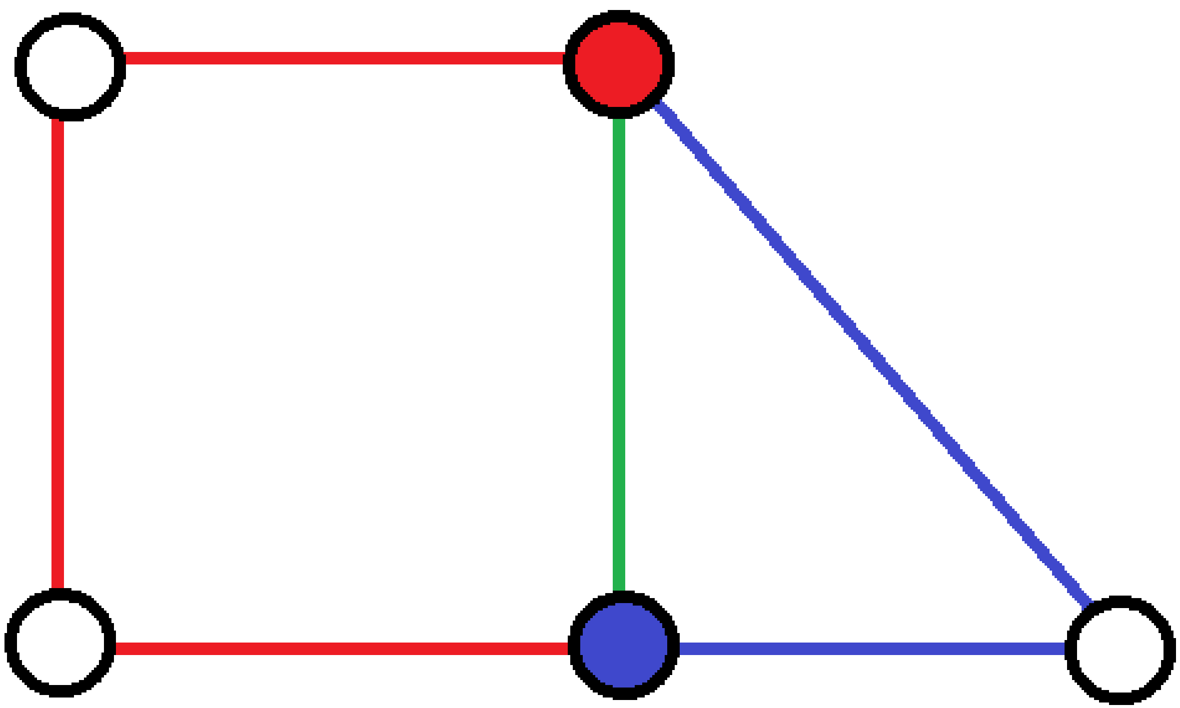




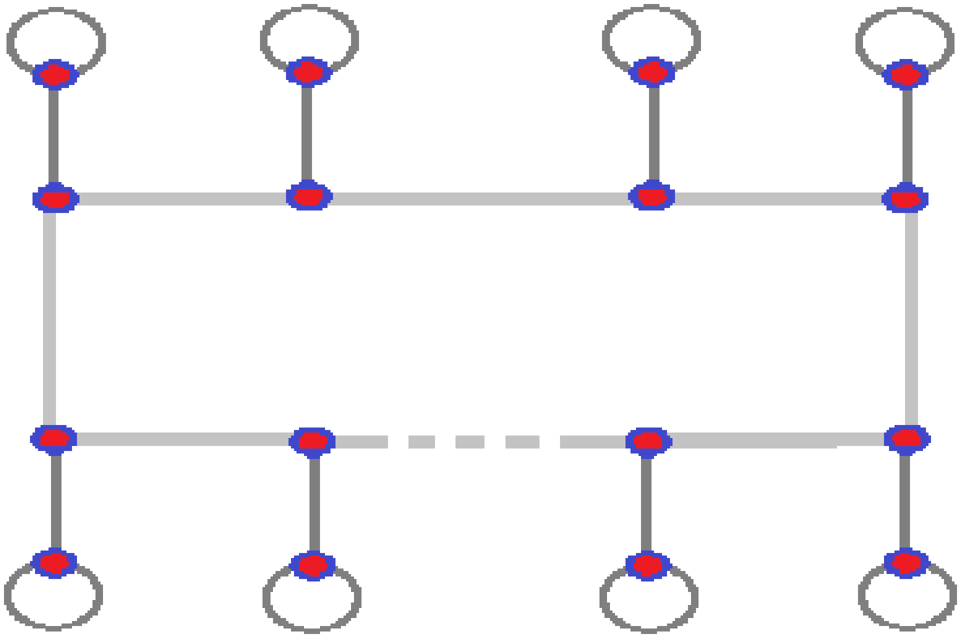
We saw this:
(3,4)-cage
(3,6)-cage

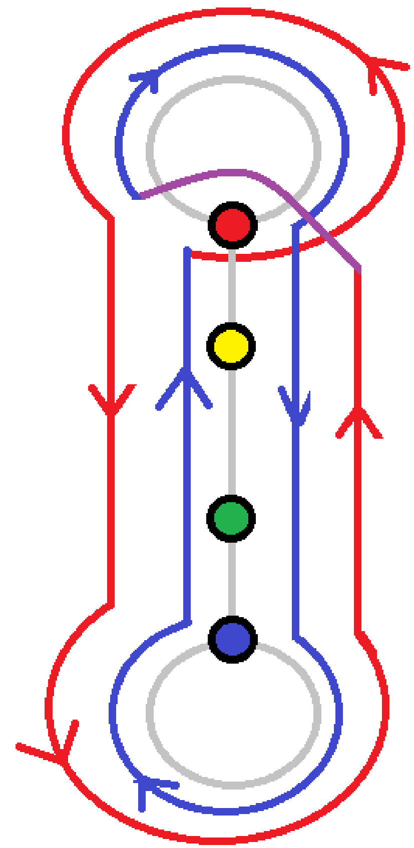
Theorem proves girth is at most 6.

Corollary:
If the underlying small graph G has this as a subgraph then the girth is at most 12.



Sunshine graphs have a bad barbell and this limits the girth to 16:





Programs use BFS to determine girth and to compute for a small graph on n vertices a lower bound on p for the girth g where the big graph has n*p vertices.

An exponential backtrack is currently being used to try and assign jumps.

Future work:

1. 3-regular cages- see Gordon Royle's page for open cases.

2. Cages: See survey by P. K. Wong.

3. Constructions for "small" graphs of large girth.

4. Backtrack to solve other problems.

5. Non-uniform voltage graphs.