The Feline Josephus Problem

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"un gatto ha sette vite"

Way back in 1968 .... from Knuth Vol. 1

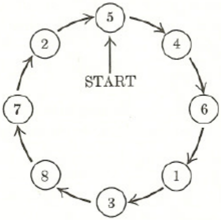


Fig. 17. Josephus' problem, $n = 8$, $m = 4$.

many of which are good programming exercises, see W. W. Rouse Ball, *Mathematical Recreations and Essays*, rev. by H. S. M. Coxeter (New York: Macmillan, 1962), Chapter 7.]

22. [*31*] "Josephus' problem." There are n men arranged in a circle. Beginning at a particular position, we count around the circle and brutally execute every mth man (the circle closing as men are decapitated). For example, the execution order when $n = 8$, $m = 4$ is 54613872, as shown in Fig. 17. Write a complete **MIX** program which prints out the order of execution when $n = 24$, $m = 11$. Try to design a clever algorithm which works at high speed when n and m are large (it may save your life). *Reference:* W. W. Rouse Ball, as in the previous exercise, pp. 32–36.

with MIX solution:

22.

```
*JOSEPHUS PROBLEM
N    EQU   24
M    EQU   11
X    ORIG  *+N
OH   ENT1  N-1            1               Set each cell to
     STZ   X+N-1          1                 number of next man
     ST1   X-1,1          N-1               in the sequence.
     DEC1  1              N-1
     J1P   *-2            N-1
     ENT1  0              1
     ENTA  1              1
1H   ENT2  M-2            N-1             (assume M > 2)
     LD1   X,1            (M-2)(N-1)      Count around
     DEC2  1              (M-2)(N-1)        the circle.
     J2P   *-2            (M-2)(N-1)
     LD2   X,1            N-1             rI1 ≡ lucky man
     LD3   X,2            N-1             rI2 ≡ doomed man
     CHAR                 N-1             rI3 ≡ next man
     STX   X,2(4:5)       N-1             Store execution number.
     NUM                  N-1
     INCA  1              N-1
     ST3   X,1            N-1             Take man from circle.
     ENT1  0,3            N-1
     CMPA  =N=            N-1
     JL    1B             N-1
     CHAR                 1               One man left
     STX   X,1(4:5)       1               (he is clobbered too).
     OUT   X(18)          1               Print answer.
     HLT                  1
     END   OB                             ▮
```

The last man is in position 11. The total time before output is $(4(N-1)(M+3)+8)u$. Is there a better method?

From the latest edition of Vol. 1

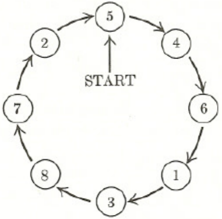


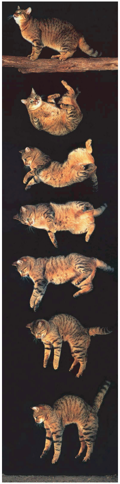
Fig. 17. Josephus' problem, $n = 8$, $m = 4$.

22. [*31*] (*The Josephus problem.*) There are n men arranged in a circle. Beginning at a particular position, we count around the circle and brutally execute every mth man; the circle closes as men die. For example, the execution order when $n = 8$ and $m = 4$ is 54613872, as shown in Fig. 17: The first man is fifth to go, the second man is fourth, etc. Write a complete MIX program that prints out the order of execution when $n = 24$, $m = 11$. Try to design a clever algorithm that works at high speed when n and m are large (it may save your life). *Reference:* W. Ahrens, *Mathematische Unterhaltungen und Spiele* **2** (Leipzig: Teubner, 1918), Chapter 15.

The problem is also featured in Concrete Mathematics

- A nice result when $k = 2$: $J(\ (10101100)_2\ ) = (1011001)_2$ (circular left shift).
- A "Bonus Problem": Suppose that Josephus finds himself in a given position j, but he has a chance to name the elimination parameter k such that every kth person is executed. Can he always save himself?

Notation for the Feline version

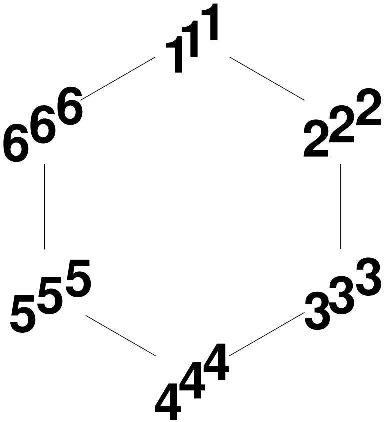


- n, the number of persons (or cats) in the circle.
- k, the *skip value*. Every k-th person (or cat) is hit.
- ℓ, if a cat is hit this many times it dies; otherwise, it is still alive and can be hit again. (Un gatto ha ℓ vite.)

Sample question: What happens if n and k are fixed and $\ell \to \infty$?
`kill`(n) for $n = 12$ and $k = 14642$ and $\ell = 1, 2, \ldots, 10$:

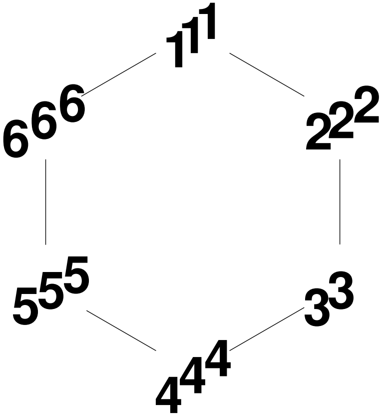
kill(n)	9	1	11	11	11	5	5	5	1	1
ℓ	1	2	3	4	5	6	7	8	9	10

Example with $n = 6$, $k = 3$ and $\ell = 3$



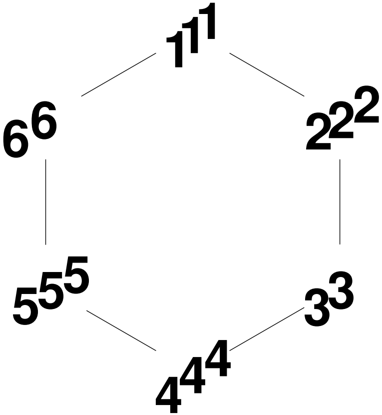
hit:

Example with $n = 6$, $k = 3$ and $\ell = 3$



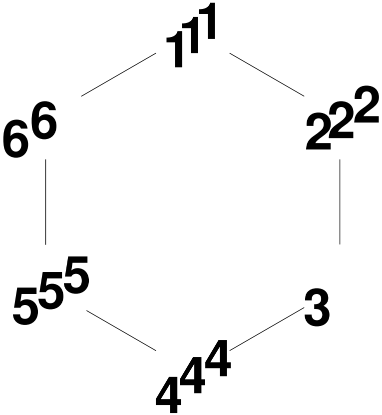
hit: 3

Example with $n = 6$, $k = 3$ and $\ell = 3$



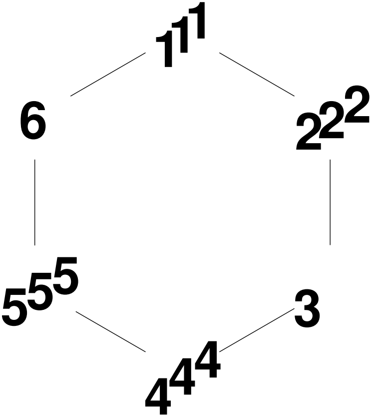
hit: 36

Example with $n = 6$, $k = 3$ and $\ell = 3$



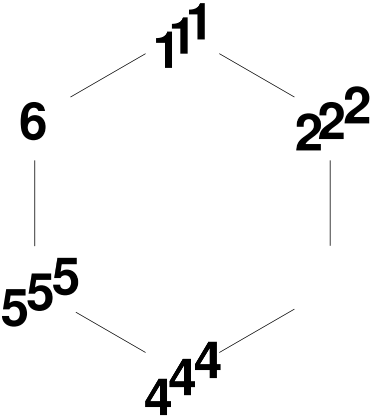
hit: 363

Example with $n = 6$, $k = 3$ and $\ell = 3$



hit: 3636

Example with $n = 6$, $k = 3$ and $\ell = 3$



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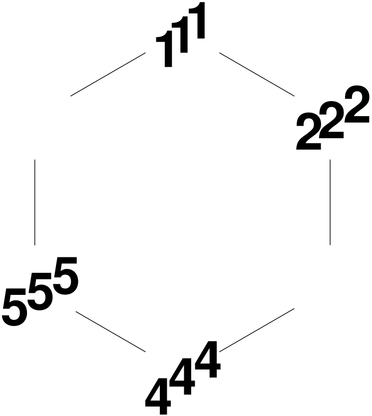
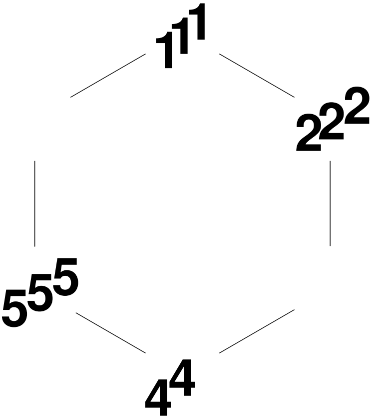
Example with $n = 6$, $k = 3$ and $\ell = 3$



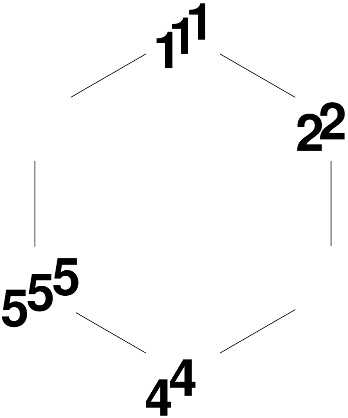
hit: 363636

Example with $n = 6$, $k = 3$ and $\ell = 3$



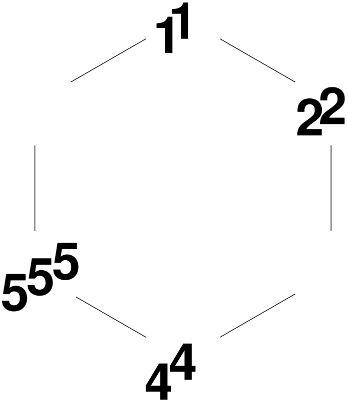
hit: $(36)^3 4$

Example with $n = 6$, $k = 3$ and $\ell = 3$



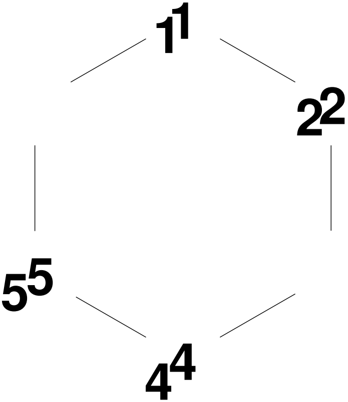
hit: $(36)^3 42$

Example with $n = 6$, $k = 3$ and $\ell = 3$



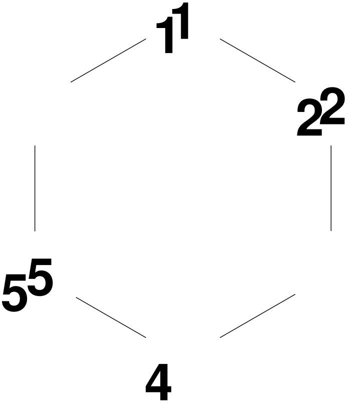
hit: $(36)^3421$

Example with $n = 6$, $k = 3$ and $\ell = 3$



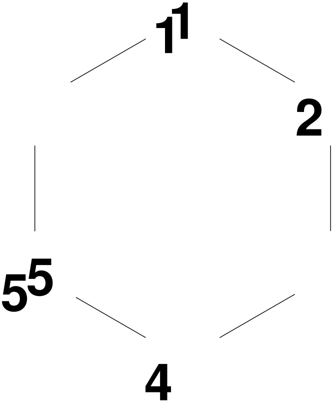
hit: $(36)^3 4215$

Example with $n = 6$, $k = 3$ and $\ell = 3$



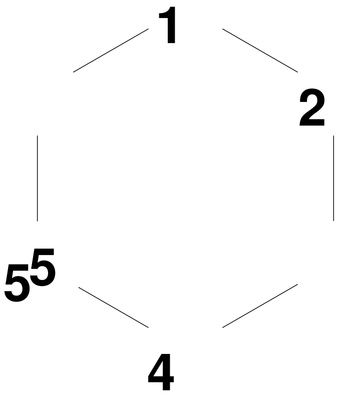
hit: $(36)^3 42154$

Example with $n = 6$, $k = 3$ and $\ell = 3$



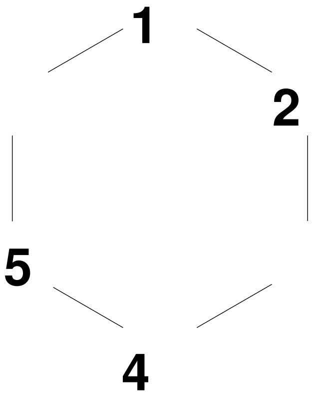
hit: $(36)^3421542$

Example with $n = 6$, $k = 3$ and $\ell = 3$



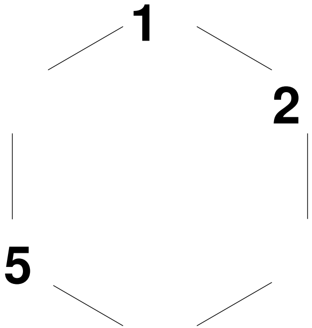
hit: $(36)^3 4215421$

Example with $n = 6$, $k = 3$ and $\ell = 3$



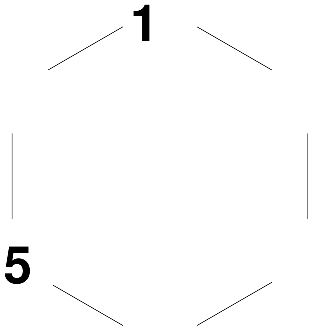
hit: $(36)^3 42154215$

Example with $n = 6$, $k = 3$ and $\ell = 3$



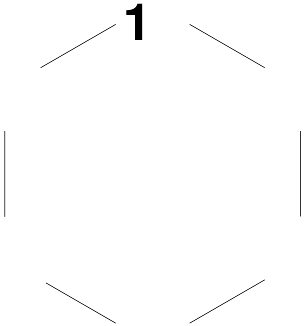
hit: $(36)^3(4215)^2 4$

Example with $n = 6$, $k = 3$ and $\ell = 3$



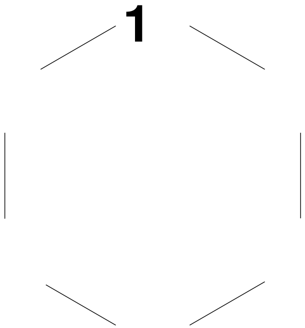
hit: $(36)^3(4215)^2 42$

Example with $n = 6$, $k = 3$ and $\ell = 3$



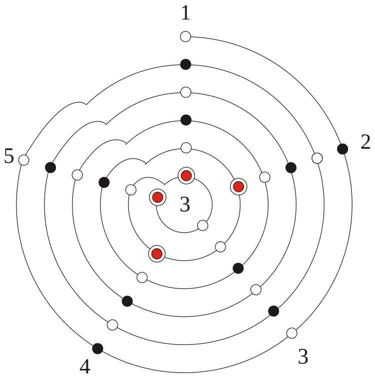
hit: $(36)^3(4215)^2 425$

Example with $n = 6$, $k = 3$ and $\ell = 3$



hit: $(36)^3(4215)^2 4251$

More terminology (here $n = 5$, $k = 2$ and $\ell = 3$)



$$\texttt{hit}(1..15) = \underbrace{2,\ 4,\ 1,\ 3,\ 5,\ 2,\ 4,\ 1,\ 3,\ 5,\ 2,}_{\texttt{round}(1)}\ \underbrace{4,\ 1,\ 5,\ 3,}_{\texttt{round}(2..5)}$$

$$\texttt{kill}(1..5) = 2, 4, 1, 5, 3.$$

Two Observations

Cats $=$ humans when $n \perp k$ (i.e., when $\gcd(n, k) = 1$) observation.

- When $n \perp k$, the kill sequence, τ, is the same for all values of ℓ.
- The hit sequence will have the form $\pi^{\ell-1}\tau$, where π and τ are both permutations of $1, 2, \ldots, n$.

Two Observations

Useless spiral observation:

- In round r there is a "useless spiral" if $k > n - r + 1$.
- So in that round we can use $k \bmod (n - r + 1)$ instead of k.
- The hit sequence is the same using k or $k \bmod \operatorname{lcm}\{1, 2, \ldots, n\}$.

Our two theorems

If ℓ is large enough, then $\texttt{kill}(n)$ is fixed.

Theorem

Given fixed values of n and k, the last surviving element $\texttt{kill}(n)$ is constant in the feline Josephus problem for all $\ell \geq F_{n+2}$, where F_n is the n-th Fibonacci number.

Josephus can always save himself *if* he is allowed to choose k.

Theorem

Suppose n and j are fixed and satisfy $1 \leq j \leq n$, and ℓ is an arbitrary positive integer. There exists a value of k such that $\texttt{kill}(n) = j$.

Behavior for large ℓ

Example with $n = 12$, $k = 14642$ (and $r' = 14642 \bmod (n-r)$).

r	r'	1	2	3	4	5	6	7	8	9	10	11	12
0	2	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ
1	1	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1	ℓ	1
2	2	ℓ	.	$\ell-1$	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1
3	8	ℓ	.	$\ell-1$	.	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1
4	2	$\ell-1$	.	$\ell-2$	.	ℓ	.	ℓ	1	ℓ	1	ℓ	**0**
5	5	$\ell-1$	.	**0**	.	ℓ	.	3	1	3	1	3	.
6	2	$\ell-1$	.	.	.	ℓ	.	3	1	3	**0**	3	.
7	2	$\ell-4$	.	.	.	ℓ	.	**0**	1	1	.	3	.
8	2	$\ell-4$	.	.	.	ℓ	.	.	1	**0**	.	3	.
9	2	$\ell-5$	.	.	.	ℓ	.	.	**0**	.	.	3	.
10	0	$\ell-8$	.	.	.	$\ell-2$	.	.	.	.	.	**0**	.
11	0	$\ell-8$	.	.	.	**0**	.	.	.	.	.	.	.
12		**0**	.	.	.	.	.	.	.	.	.	.	.

$\texttt{round}(1) = (2, 4, 6, 8, 10, 12)^{\ell-1}, 2$

Behavior for large ℓ

Example with $n = 12$, $k = 14642$ (and $r' = 14642 \bmod (n - r)$).

r	r'	1	2	3	4	5	6	7	8	9	10	11	12
0	2	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ
1	1	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1	ℓ	1
2	2	ℓ	.	$\ell-1$	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1
3	8	ℓ	.	$\ell-1$	.	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1
4	2	$\ell-1$	.	$\ell-2$	.	ℓ	.	ℓ	1	ℓ	1	ℓ	**0**
5	5	$\ell-1$	.	**0**	.	ℓ	.	3	1	3	1	3	.
6	2	$\ell-1$	.	.	.	ℓ	.	3	1	3	**0**	3	.
7	2	$\ell-4$	.	.	.	ℓ	.	**0**	1	1	.	3	.
8	2	$\ell-4$	.	.	.	ℓ	.	.	1	**0**	.	3	.
9	2	$\ell-5$	.	.	.	ℓ	.	.	**0**	.	.	3	.
10	0	$\ell-8$	.	.	.	$\ell-2$	.	.	.	.	.	**0**	.
11	0	$\ell-8$	.	.	.	**0**	.	.	.	.	.	.	.
12		**0**	.	.	.	.	.	.	.	.	.	.	.

$\mathtt{round}(1) = (2,4,6,8,10,12)^{\ell-1}, 2 \qquad \mathtt{round}(2) = 3, 4$

Behavior for large ℓ

Example with $n = 12$, $k = 14642$ (and $r' = 14642 \bmod (n-r)$).

r	r'	1	2	3	4	5	6	7	8	9	10	11	12
0	2	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ
1	1	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1	ℓ	1
2	2	ℓ	.	$\ell-1$	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1
3	8	ℓ	.	$\ell-1$	.	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1
4	2	$\ell-1$	.	$\ell-2$	.	ℓ	.	ℓ	1	ℓ	1	ℓ	**0**
5	5	$\ell-1$	.	**0**	.	ℓ	.	3	1	3	1	3	.
6	2	$\ell-1$	.	.	.	ℓ	.	3	1	3	**0**	3	.
7	2	$\ell-4$	.	.	.	ℓ	.	**0**	1	1	.	3	.
8	2	$\ell-4$	.	.	.	ℓ	.	.	1	**0**	.	3	.
9	2	$\ell-5$	.	.	.	ℓ	.	.	**0**	.	.	3	.
10	0	$\ell-8$	.	.	.	$\ell-2$	.	.	.	.	.	**0**	.
11	0	$\ell-8$	.	.	.	**0**	.	.	.	.	.	.	.
12		**0**	.	.	.	.	.	.	.	.	.	.	.

$\texttt{round}(1) = (2,4,6,8,10,12)^{\ell-1}, 2$ $\quad$ $\texttt{round}(2) = 3,4$

$\texttt{round}(3) = 6$

Behavior for large ℓ

Example with $n = 12$, $k = 14642$ (and $r' = 14642 \bmod (n{-}r)$).

r	r'	1	2	3	4	5	6	7	8	9	10	11	12
0	2	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ
1	1	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1	ℓ	1
2	2	ℓ	.	$\ell-1$	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1
3	8	ℓ	.	$\ell-1$	.	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1
4	2	$\ell-1$	.	$\ell-2$	.	ℓ	.	ℓ	1	ℓ	1	ℓ	**0**
5	5	$\ell-1$	.	**0**	.	ℓ	.	3	1	3	1	3	.
6	2	$\ell-1$	.	.	.	ℓ	.	3	1	3	**0**	3	.
7	2	$\ell-4$	.	.	.	ℓ	.	**0**	1	1	.	3	.
8	2	$\ell-4$	.	.	.	ℓ	.	.	1	**0**	.	3	.
9	2	$\ell-5$	.	.	.	ℓ	.	.	**0**	.	.	3	.
10	0	$\ell-8$	.	.	.	$\ell-2$	.	.	.	.	.	**0**	.
11	0	$\ell-8$	.	.	.	**0**	.	.	.	.	.	.	.
12		**0**	.	.	.	.	.	.	.	.	.	.	.

$\mathtt{round}(1) = (2,4,6,8,10,12)^{\ell-1}, 2$ $\quad$ $\mathtt{round}(2) = 3, 4$

$\mathtt{round}(3) = 6$ $\quad$ $\mathtt{round}(4) = 3, 1, 12$

Behavior for large ℓ

Example with $n = 12$, $k = 14642$ (and $r' = 14642 \bmod (n-r)$).

r	r'	1	2	3	4	5	6	7	8	9	10	11	12
0	2	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ
1	1	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1	ℓ	1
2	2	ℓ	.	$\ell-1$	**0**	ℓ	1	ℓ	1	ℓ	1	ℓ	1
3	8	ℓ	.	$\ell-1$	.	ℓ	**0**	ℓ	1	ℓ	1	ℓ	1
4	2	$\ell-1$	.	$\ell-2$	.	ℓ	.	ℓ	1	ℓ	1	ℓ	**0**
5	5	$\ell-1$	.	**0**	.	ℓ	.	3	1	3	1	3	.
6	2	$\ell-1$	.	.	.	ℓ	.	3	1	3	**0**	3	.
7	2	$\ell-4$	.	.	.	ℓ	.	**0**	1	1	.	3	.
8	2	$\ell-4$	.	.	.	ℓ	.	.	1	**0**	.	3	.
9	2	$\ell-5$	.	.	.	ℓ	.	.	**0**	.	.	3	.
10	0	$\ell-8$	.	.	.	$\ell-2$	.	.	.	.	.	**0**	.
11	0	$\ell-8$	.	.	.	**0**	.	.	.	.	.	.	.
12		**0**	.	.	.	.	.	.	.	.	.	.	.

$\mathtt{round}(1) = (2,4,6,8,10,12)^{\ell-1}, 2$ $\quad$ $\mathtt{round}(2) = 3, 4$

$\mathtt{round}(3) = 6$ $\quad$ $\mathtt{round}(4) = 3, 1, 12$

Behavior for large ℓ cont.

If in round r (assuming that $a < b$)

$$\ell_r(i) \in \{1, 2, \ldots, a\} \cup \{\ell, \ell - 1, \ldots, \ell - b\},$$

then in round $r + 1$

$$\ell_{r+1}(i) \in \underbrace{\{1, 2, \ldots, b + 1\}}_{\ell - b \;\to\; 0} \cup \underbrace{\{\ell, \ell - 1, \ldots, \ell - (a + b)\}}_{a \;\to\; 0}$$

The successive values of a and b satisfy a Fibonacci-like recursion, and the initial conditions imply that

$$a = F_r \text{ and } b = F_{r+1} - 1.$$

Josephus can save himself if he can pick k

- If $j = n$ then take $k = 1$;
  hit sequence is $(1, 2, \ldots, n)^\ell$.
- If $j = 1$ then take $k = \mathrm{lcm}\{1, 2, \ldots, n\}$;
  hit sequence is $n^\ell, (n-1)^\ell, \ldots, 2^\ell, 1^\ell$.

Let p be a prime such that $n/2 < p < n$.
Let $\mathcal{P} = \{2, 3, \ldots, n\} - \{p\}$.

- Case A: $n/2 \le j < n$. The hit sequence

$$(1, 2, \ldots, n)^{\ell-1}, 1, 2, \ldots, n{-}p,$$
$$j{+}1, j{+}2, \ldots, n, n{-}p{+}1, n{-}p{+}2, \ldots, j$$

  may be obtained by taking

$$k \equiv 1 \bmod \mathrm{lcm}\,\mathcal{P}$$
$$k \equiv j + 1 - n \bmod p.$$

Josephus can save himself if he can pick k

- If $j = n$ then take $k = 1$;
  hit sequence is $(1, 2, \ldots, n)^\ell$.
- If $j = 1$ then take $k = \operatorname{lcm}\{1, 2, \ldots, n\}$;
  hit sequence is $n^\ell, (n-1)^\ell, \ldots, 2^\ell, 1^\ell$.

Let p be a prime such that $n/2 < p < n$.
Let $\mathcal{P} = \{2, 3, \ldots, n\} - \{p\}$.

- Case B: $1 < j < n/2$. The hit sequence

$$n^\ell, (n{-}1)^\ell, \ldots, (p{+}1)^\ell, \pi^{\ell-1},$$
$$j{-}1, j{-}2, \ldots, 1, p, p{-}1, \ldots, j,$$

  where π is a permutation of $\{1, 2, \ldots, p\}$ beginning with $j-1$, may be obtained by taking

$$k \equiv 0 \bmod \operatorname{lcm} \mathcal{P}$$
$$k \equiv j-1 \bmod p.$$

Open Problems

- What is the complexity of determining kill(n), the kill() permutation, etc.?
- All hits at the end: $(n=6,\ k=10,\ \ell \geq 2)$:

$$(4\ 2\ 6)^{\ell-1}\ 4,\ 3^{\ell},\ 6,\ 1\ 2,\ 1^{\ell-1},\ \textcolor{red}{5^{\ell}}.$$

  Another interesting example (the survivor is not the last hit for the first time): $(n=12,\ k=14642,\ \ell>8)$

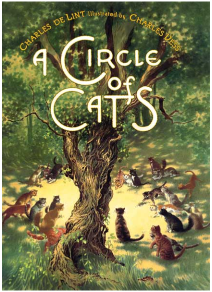
$$\ldots 10, (1,7,9)^2, 1, 7, 9, 1, 8, (1, 11, \textcolor{red}{5})^2, 1, 11, 5^{\ell-2}, 1^{\ell-8}.$$

- Better bounds on $\sigma(n)$ = largest first value of ℓ where kill(n) stabilizes.

$n=$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\sigma(n)$	1	1	1	1	1	3	3	4	6	6	6	9	9	11	14
F_{n+2}	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597

A friendly wager: 

Thanks for coming!



Any questions?