# Counting Fixed-Height Tatami Tilings

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**University of Victoria**

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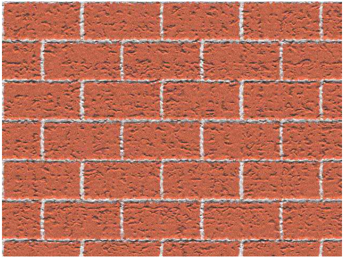

# Motivation

Don Knuth: The Art of Computer Programming Volume IV

Section 7.1.4: Binary Decision Diagrams

Solution to problem 7.1.4.214

*Conjecture:* The generating function for the number of $m \times n$ tatami tilings, when $n \geq m - 2 \geq 0$ and $m$ is even, is $(1+z)^2(z^{m-2}+z^m)/(1-z^{m-1}-z^{m+1})$.

# (Ordinary) Generating Functions

*"A generating function is a clothesline on which we hang up a sequence of numbers for display."*

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For example:

$$1 + z + z^2 + z^3 + z^4 + \dots = \frac{1}{1-z}$$

is the generating function for the sequence $1, 1, 1, 1, 1, \dots$.

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is the generating function for the sequence $1, 1, 1, 1, 1, \dots$.

In other words, $\frac{1}{1-z}$ generates $1, 1, 1, 1, 1, \dots$.

# (Ordinary) Generating Functions and Picture Arithmetic

$$A = | + \bigstar + \bigstar\bigstar + \bigstar\bigstar\bigstar + \bigstar\bigstar\bigstar\bigstar + \dots$$

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$$\begin{aligned} A &= | + \bigstar A \\ A - \bigstar A &= | \end{aligned}$$

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$$\begin{aligned} A &= | + \bigstar A \\ A - \bigstar A &= | \\ (| - \bigstar)A &= | \\ A &= \frac{|}{| - \bigstar} \end{aligned}$$

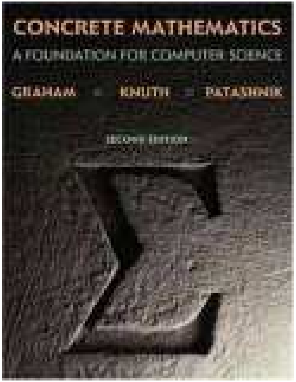
# (Ordinary) Generating Functions and Picture Arithmetic

$$\begin{aligned}
A &= | + \bigstar + \bigstar\bigstar + \bigstar\bigstar\bigstar + \bigstar\bigstar\bigstar\bigstar + \dots \\
&\leadsto z^0 + z^1 + z^1 z^1 + z^1 z^1 z^1 + z^1 z^1 z^1 z^1 + \dots \\
&\leadsto 1 + z + z^2 + z^3 + z^4 + \dots \\
A &= | + \bigstar A \\
A - \bigstar A &= | \\
(| - \bigstar)A &= | \\
A &= \frac{|}{| - \bigstar} \\
&\leadsto \frac{1}{1 - z^1}
\end{aligned}$$

# (Ordinary) Generating Functions and Picture Arithmetic

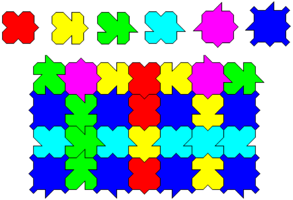
G. Polya: On Picture Writing
Am. Math Monthly Vol 63. No. 10.

Graham, Knuth, Patashnik:
Concrete Math



# Tilings in General

- There are lots of papers about tilings - of rectangles, of the plane, of other surfaces.
- Tiles can be many different shapes:



- Tiling a rectangle with dimers with no restrictions is equivalent to finding a perfect matching in a grid graph.

# What is “Tatami”?

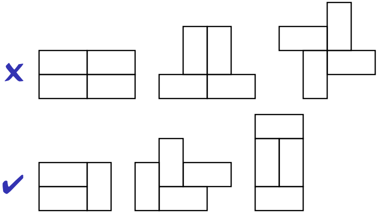
OED: A Tatami is a straw mat which is the usual floor-covering in Japan and the size of which functions as a standard unit in room measurement.

- The width to length ratio is <u>usually</u> 1:2
- The mats are laid out using an ”auspicious layout” - no four mats touch.

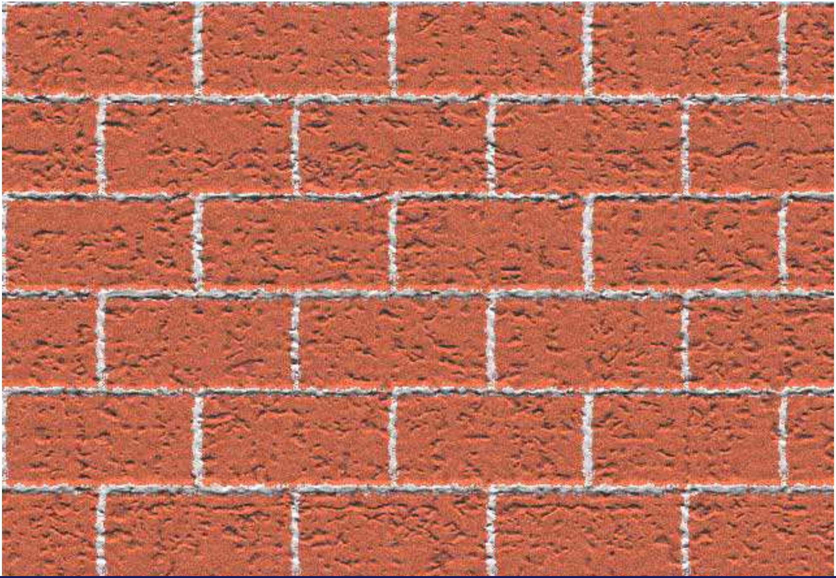


# Tatami Tiling Terminology ($T^3$)

- *dimer*: tile with width to length ratio 1:2 
- *tatami property*: no four dimers can meet at a point



# Tatami Tilings



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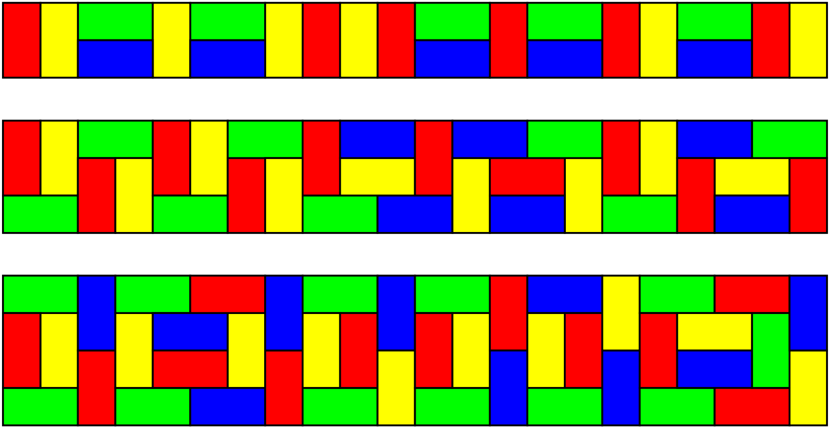
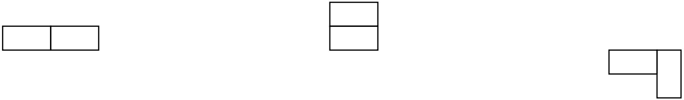


# Building Tatami Tilings



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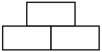

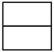

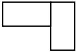

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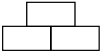

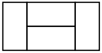

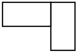

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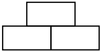

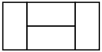

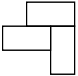

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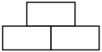
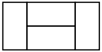
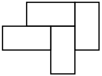

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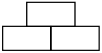

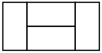

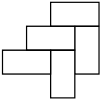

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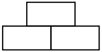

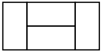

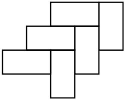

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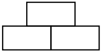

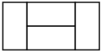

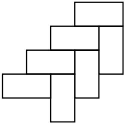

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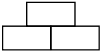

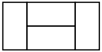

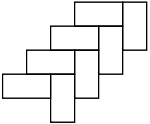

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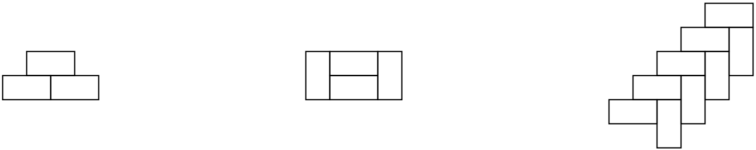

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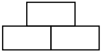
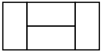
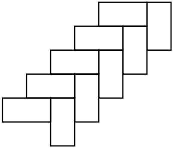
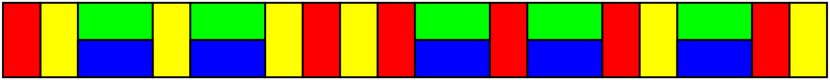
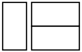


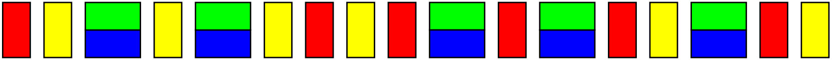




# Height 2 tatami tilings

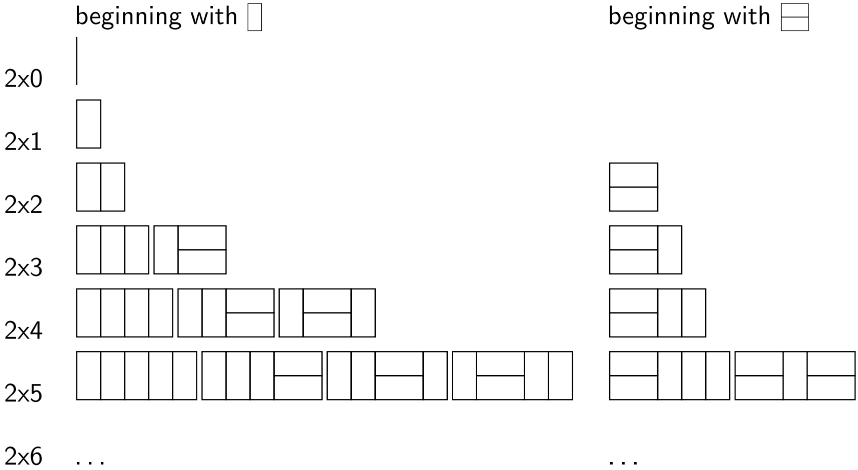


Two types of subtilings: 



# Height 2 tatami tilings

All height 2 tatami tilings:



# Height 2 tatami tilings

All height 2 tatami tilings:

beginning with ▯

2x0

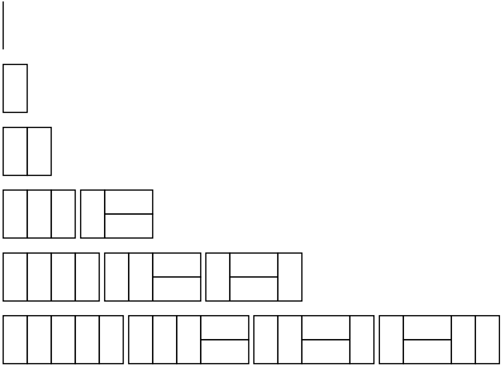
2x1

2x2

2x3

2x4

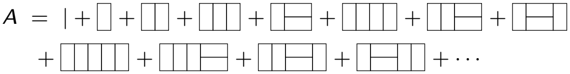
2x5



2x6 . . .

# Height 2 tatami tilings

Formal sum of height 2 tatami tilings beginning with :



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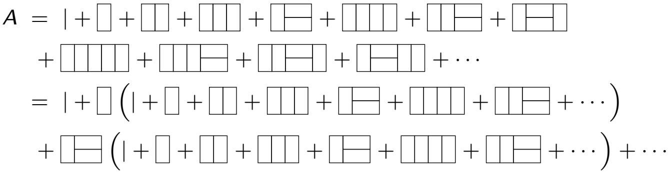

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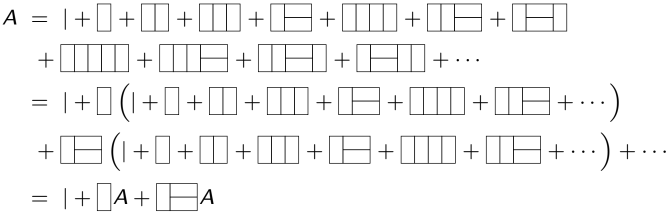

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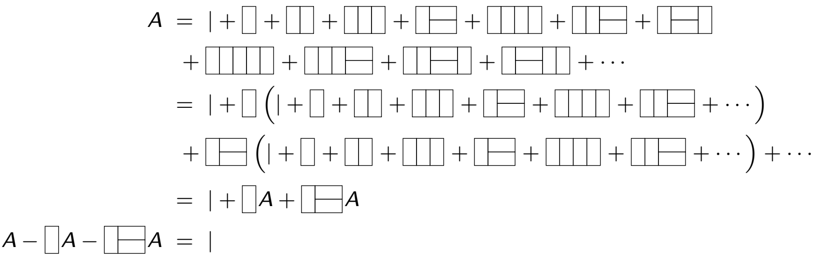
Formal sum of height 2 tatami tilings beginning with □:



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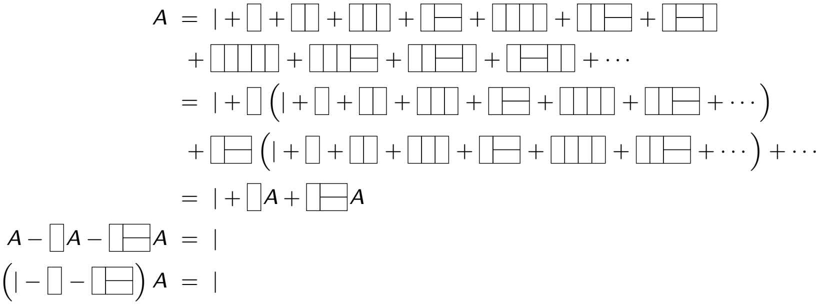

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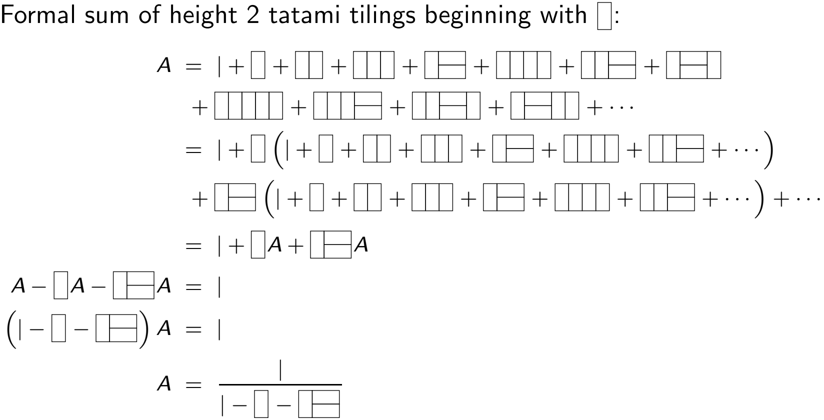
Formal sum of height 2 tatami tilings beginning with 

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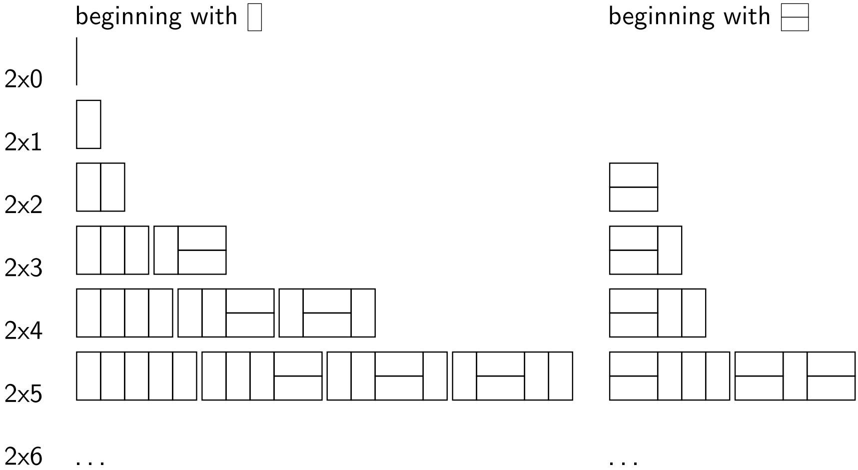
All height 2 tatami tilings:



# Height 2 tatami tilings

All height 2 tatami tilings:

beginning with 

2x0

2x1

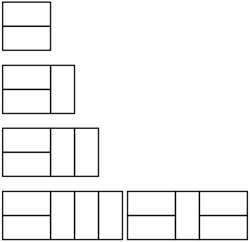
2x2

2x3

2x4

2x5

2x6



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# Height 2 tatami tilings

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beginning with ⊟

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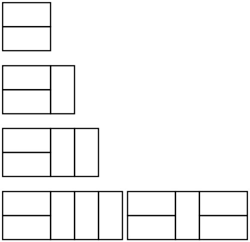

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$= ⊟A$

# Height 2 tatami tilings

Formal sum of all height 2 tilings:

# Height 2 tatami tilings

Formal sum of all height 2 tilings:

$T_2 = A +$  $A$

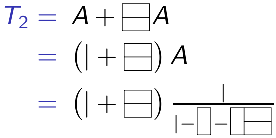
# Height 2 tatami tilings

Formal sum of all height 2 tilings:

$$\begin{aligned} T_2 &= A + \boxminus A \\ &= \left( | + \boxminus \right) A \end{aligned}$$

# Height 2 tatami tilings

Formal sum of all height 2 tilings:



# Height 2 tatami tilings

Formal sum of all height 2 tilings:

$$
\begin{aligned}
T_2 &= A + \boxminus A \\
&= \left(| + \boxminus\right) A \\
&= \left(| + \boxminus\right) \frac{|}{| - \square - \square\boxminus}
\end{aligned}
$$

Replace tilings in sum by $z^{width}$:

# Height 2 tatami tilings

Formal sum of all height 2 tilings:

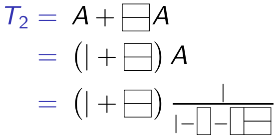
$$T_2 = A + \boxminus A$$
$$= \left(| + \boxminus\right) A$$
$$= \left(| + \boxminus\right) \frac{|}{| - \square - \square\boxminus}$$

Replace tilings in sum by $z^{width}$:

$$T_2(z) = \left(1 + z^2\right) \frac{1}{1 - z - z^3}$$

# Height 2 tatami tilings

Formal sum of all height 2 tilings:



Replace tilings in sum by $z^{width}$:

$$T_2(z) = \left(1 + z^2\right) \frac{1}{1-z-z^3}$$
$$= \frac{1+z^2}{1-z-z^3}$$

# Height 2 tatami tilings

Formal sum of all height 2 tilings:

$$\begin{aligned} T_2 &= A + \boxminus A \\ &= \left( | + \boxminus \right) A \\ &= \left( | + \boxminus \right) \frac{|}{| - \square - \square\boxminus} \end{aligned}$$

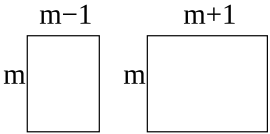
Replace tilings in sum by $z^{width}$:

$$\begin{aligned} T_2(z) &= \left(1 + z^2\right) \frac{1}{1-z-z^3} \\ &= \frac{1+z^2}{1-z-z^3} \\ &= 1+z+2z^2+3z^3+4z^4+6z^5+9z^6+13z^7+19z^8+28z^9+\cdots \end{aligned}$$

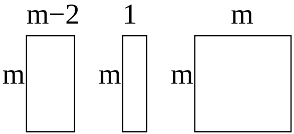
# Structure of height *m* Tatami Tilings

Theorem (Dean Hickerson):
Tatami tilings of rectangles with width ≥ height consist of some combination of tatami tilings of certain smaller rectangles.

- For odd-height rectangles



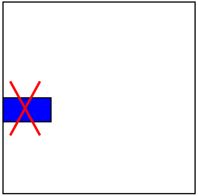

- For even-height rectangles (height ≥ 4)




http://www2.research.att.com/~njas/sequences/a068920.txt

# Structure of height $m$ Tatami Tilings

There can be no horizontal tile touching the left edge, except at the top and/or bottom of the rectangle.



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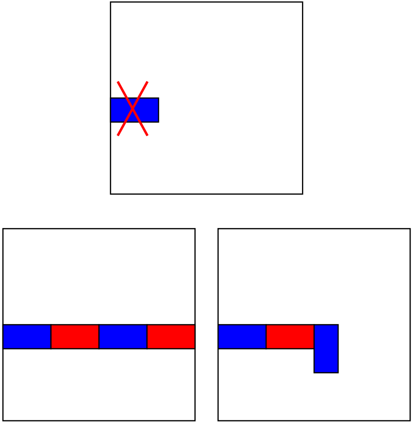

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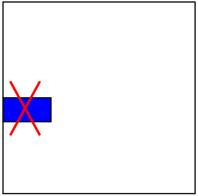

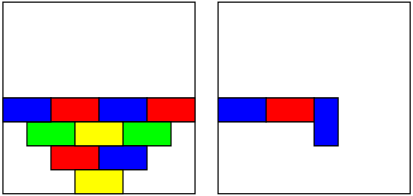

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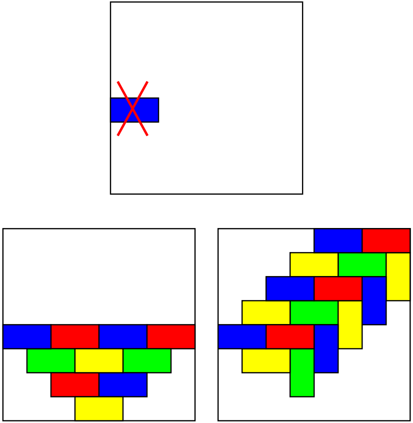

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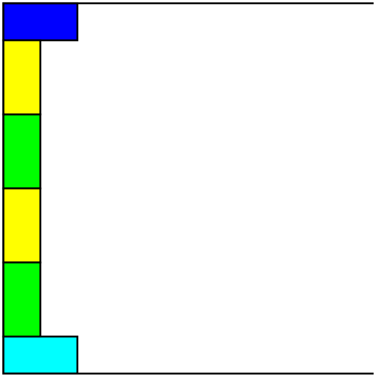

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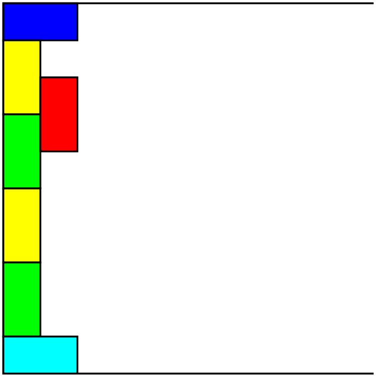

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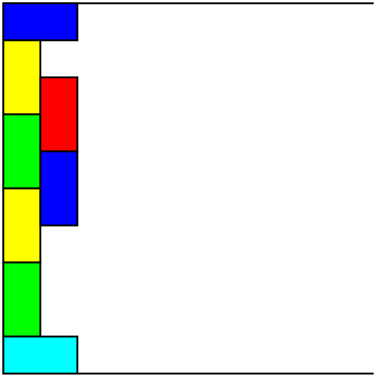

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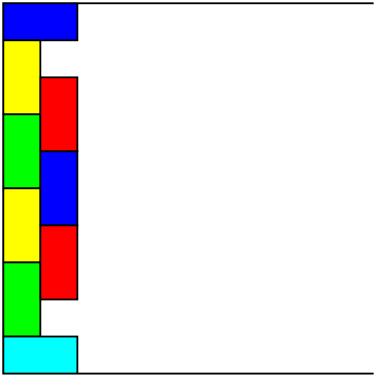

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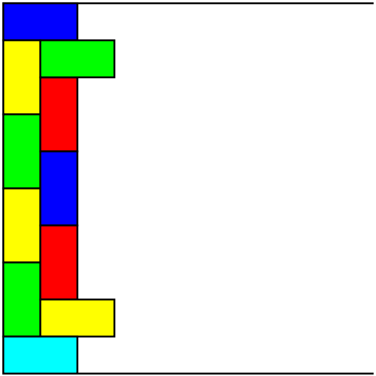

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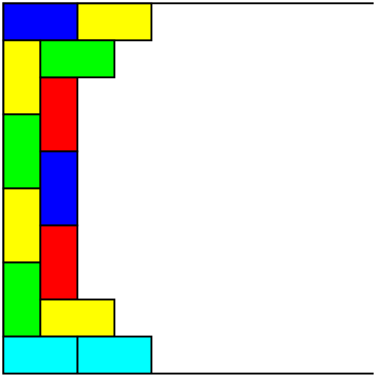

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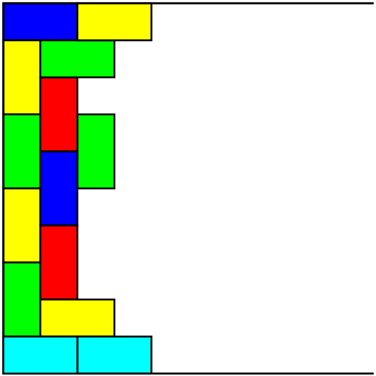

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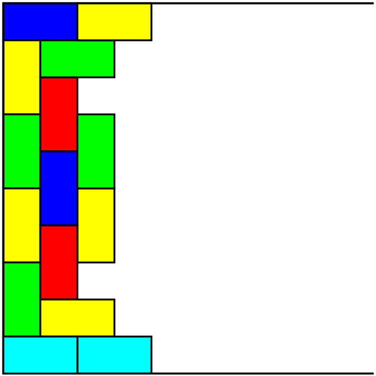

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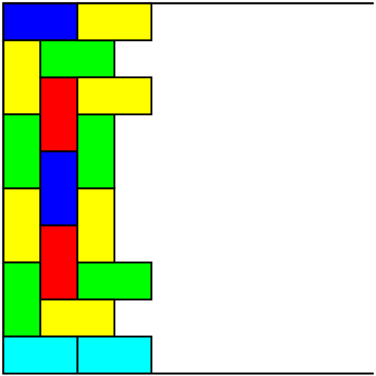

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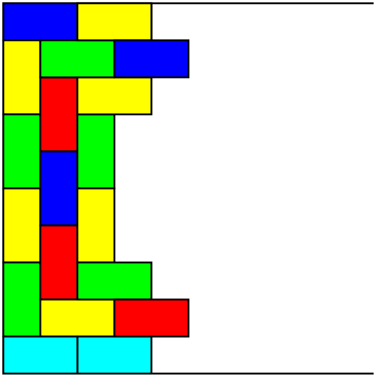

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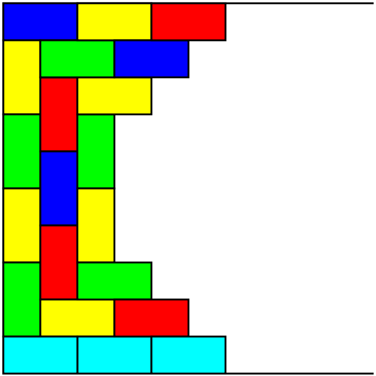

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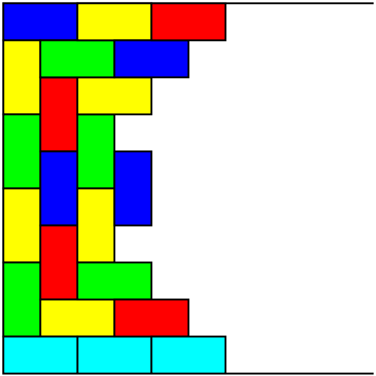

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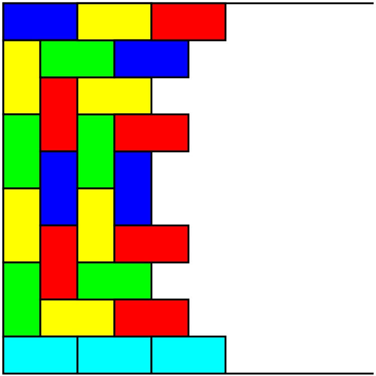

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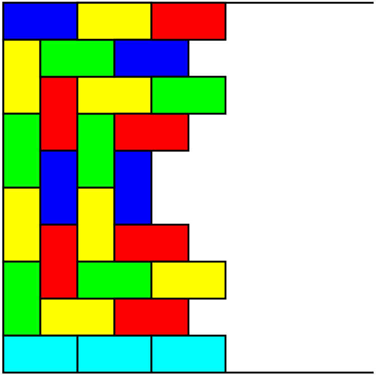

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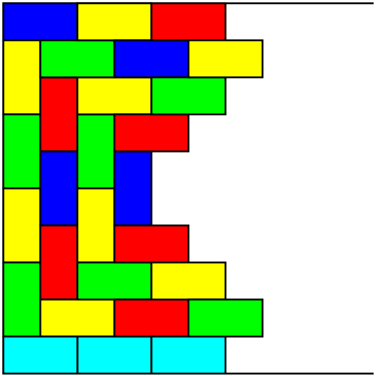

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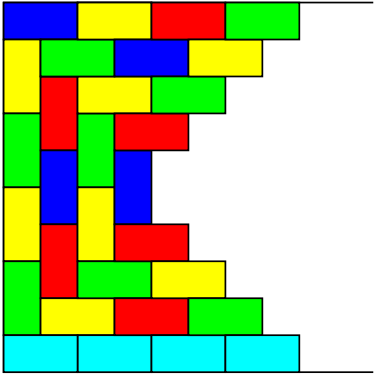

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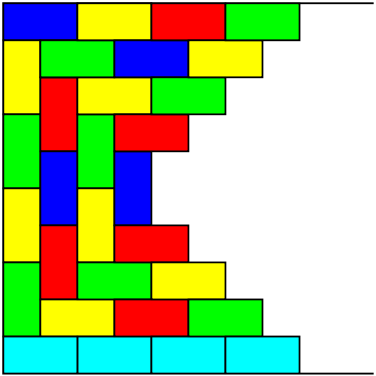

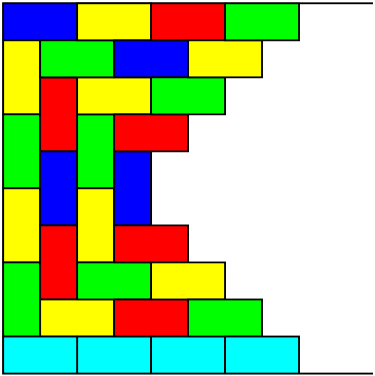

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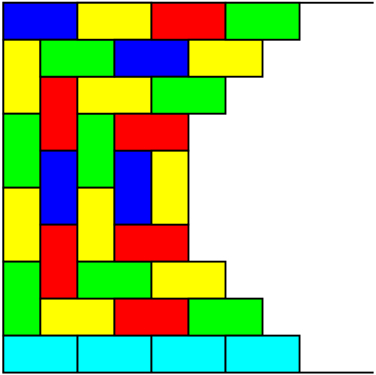

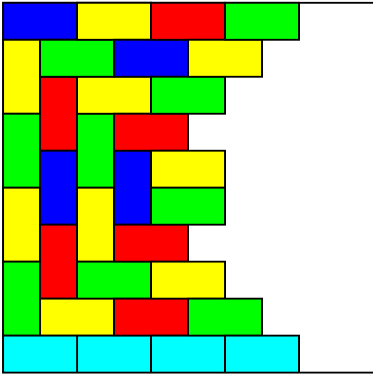

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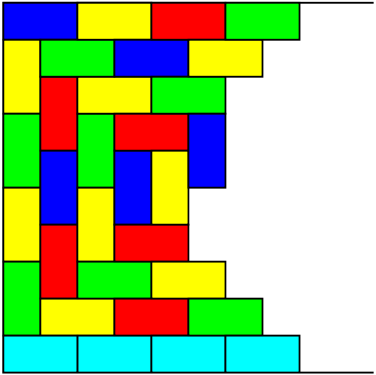

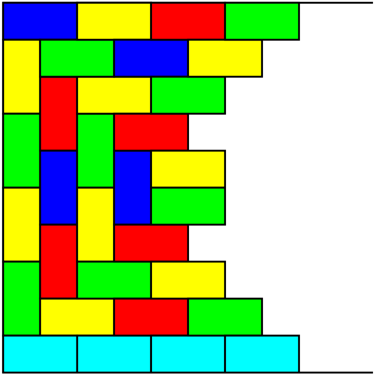

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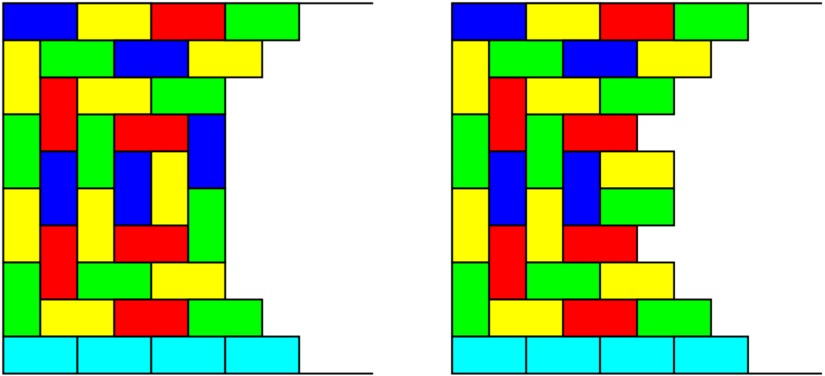

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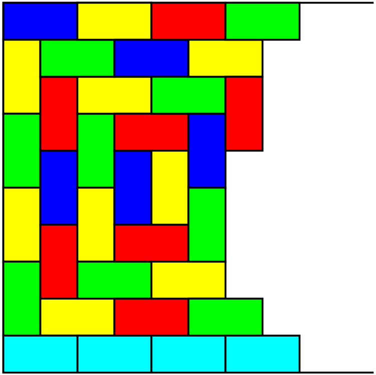

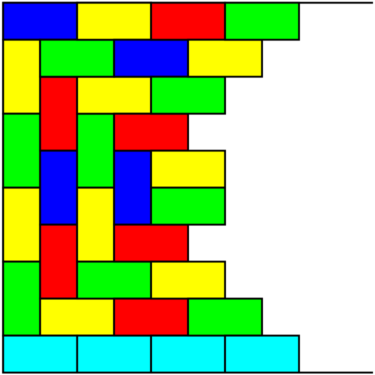

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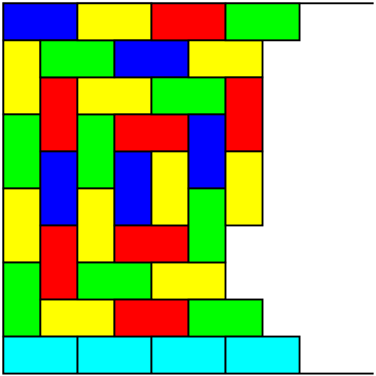

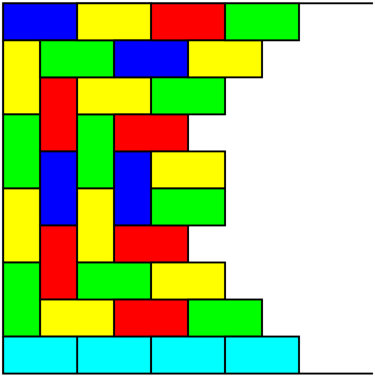

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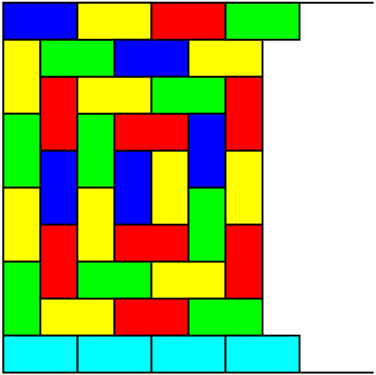

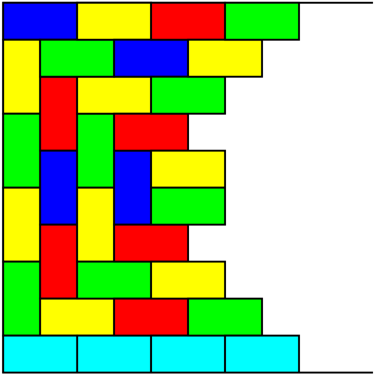

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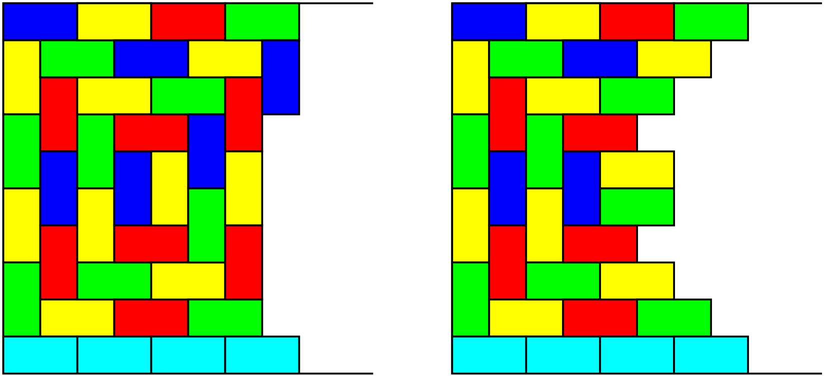

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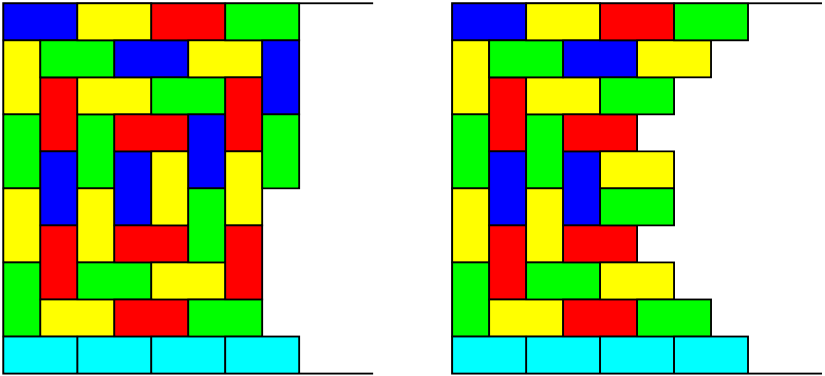

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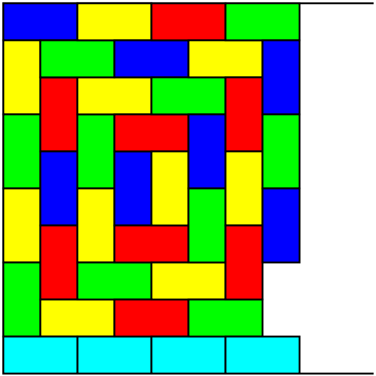

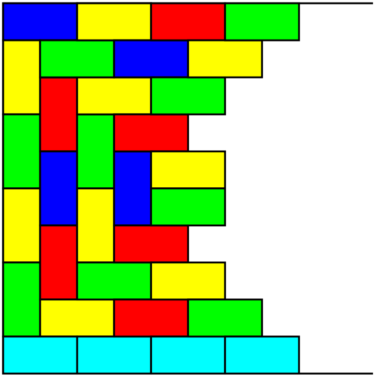

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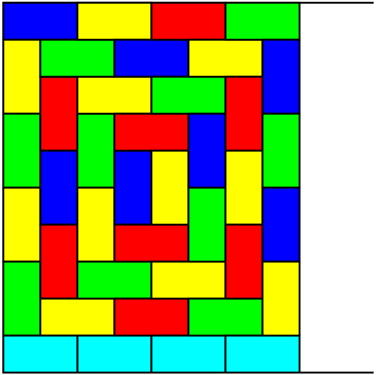

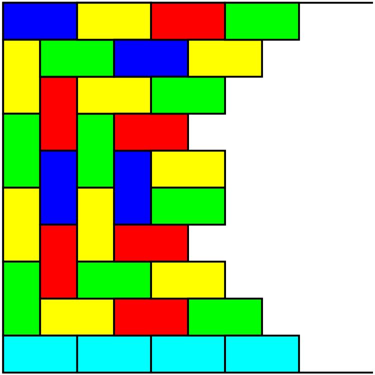

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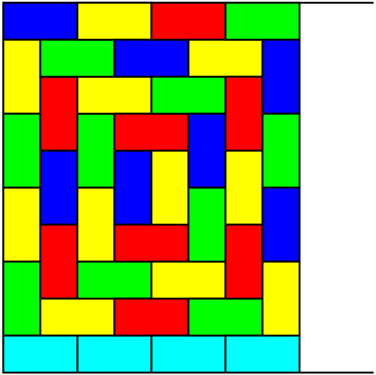

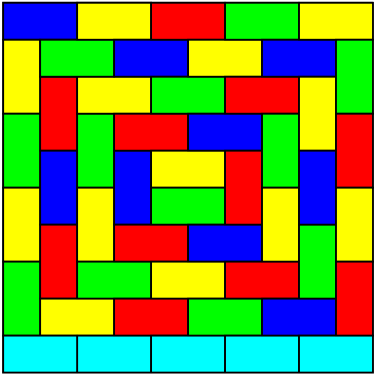

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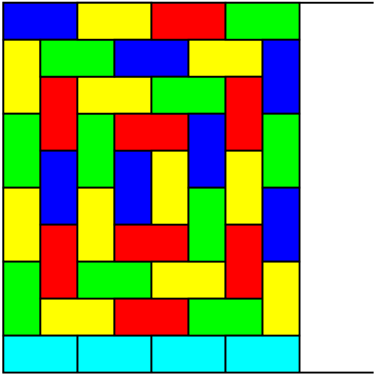

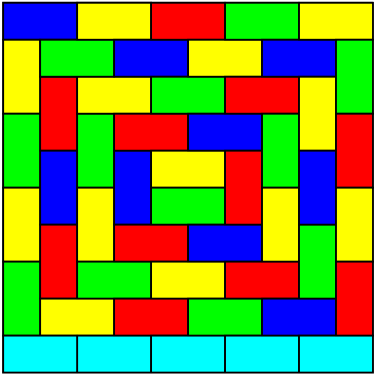

odd: m x m-1
even: m x m-2

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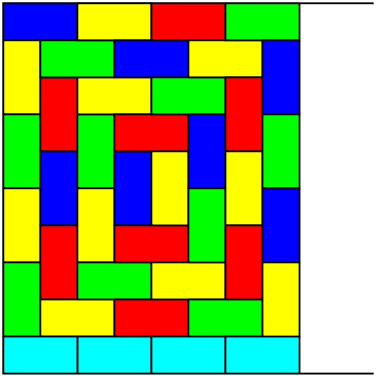

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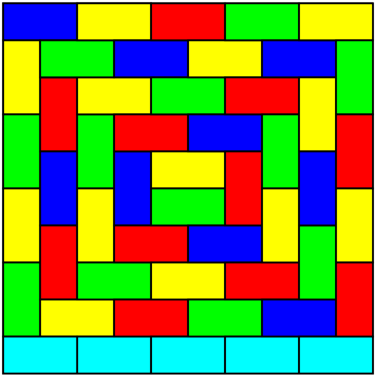
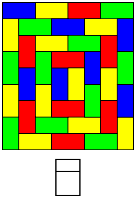
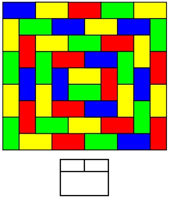
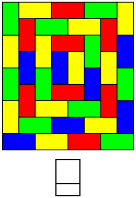


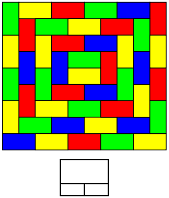
odd: m x m+1
even: m x m

# Subtiling Options for height $m$ (odd)

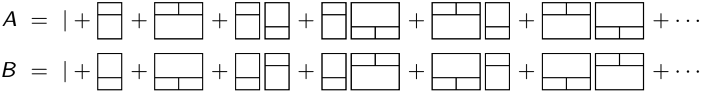








# All height $m$ tilings (odd)




$$T_m = A + B - |$$

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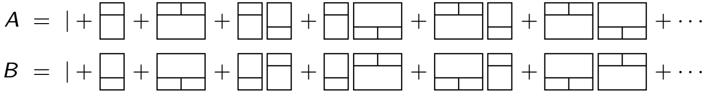

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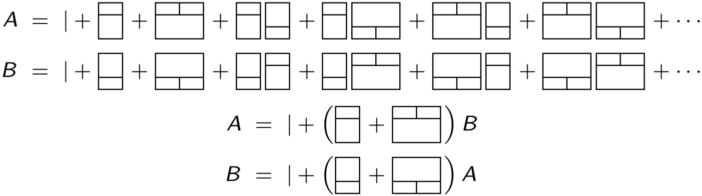

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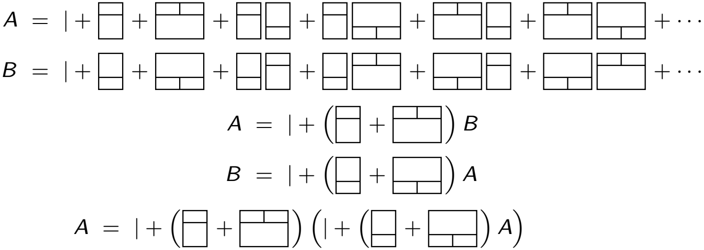

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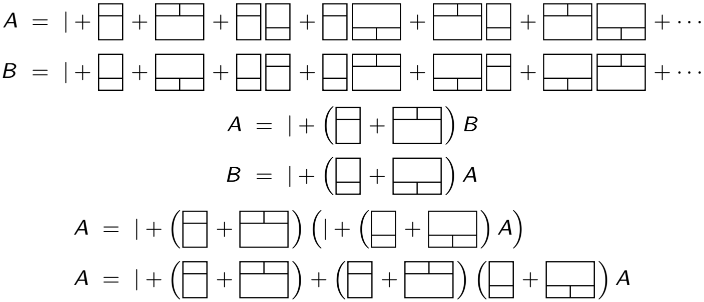

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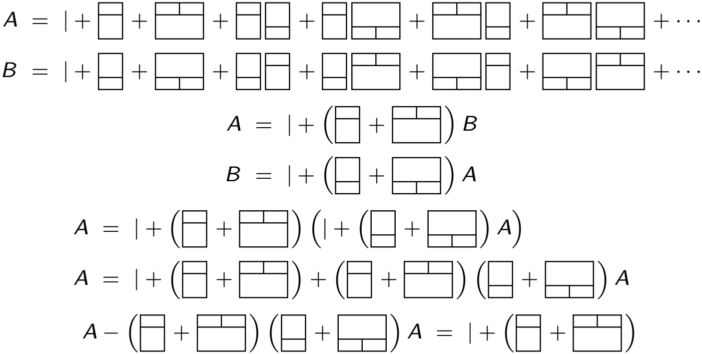

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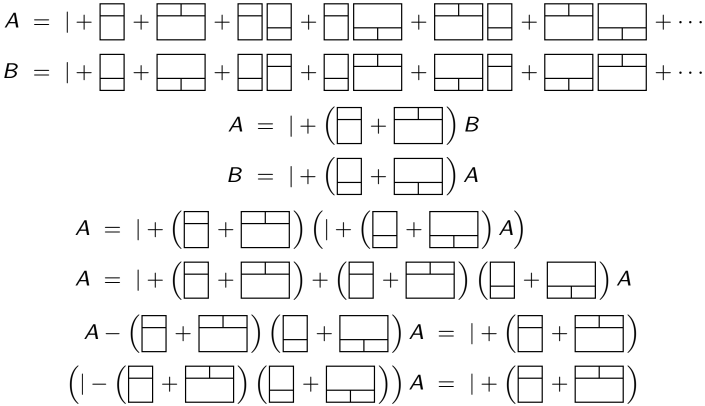

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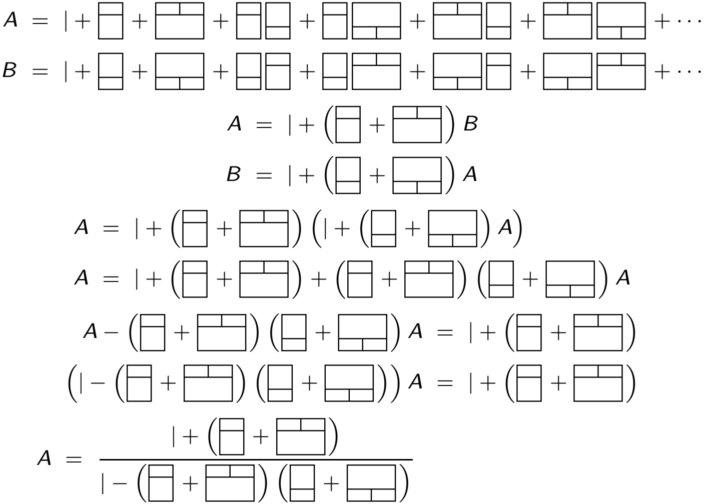

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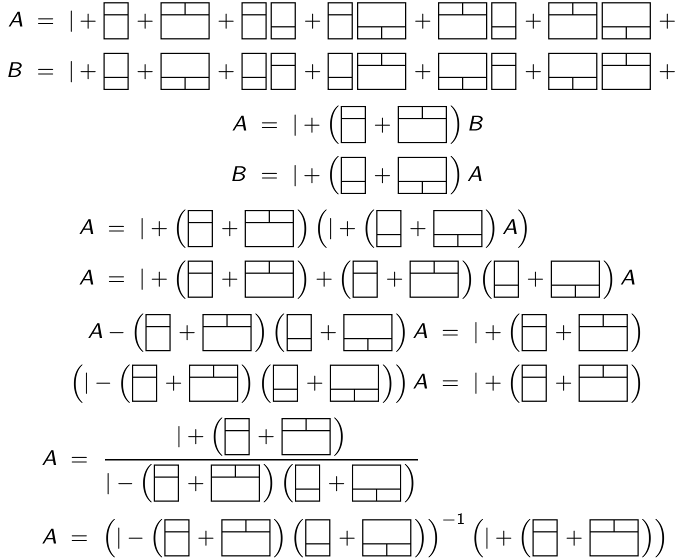

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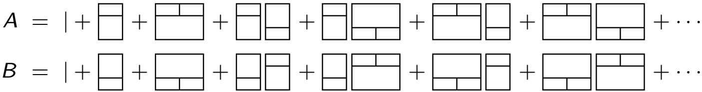

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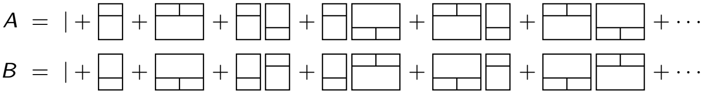


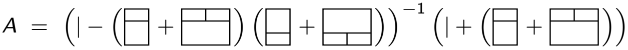

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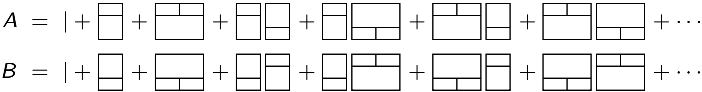


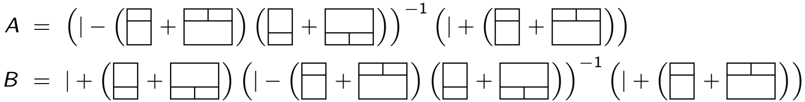

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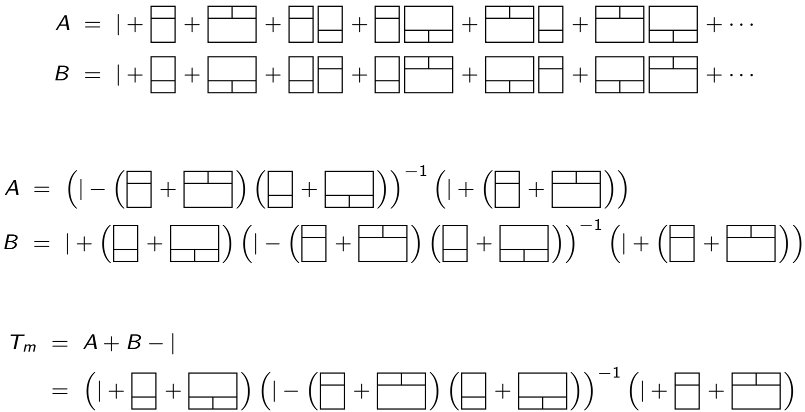
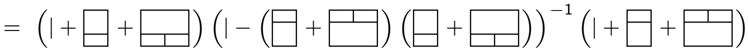


# All height $m$ tilings (odd)

$$T_m = A + B - |$$



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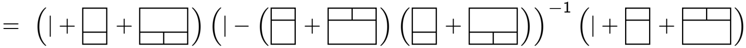
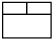
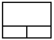
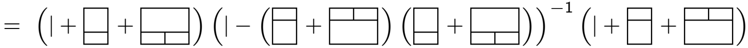


Substitute:

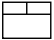
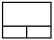
- 1 for $|$,
- $z^{m-1}$ for  and , and
- $z^{m+1}$ for  and 

# All height $m$ tilings (odd)

$$T_m = A + B - |$$



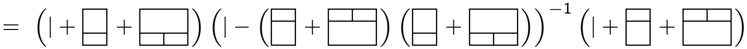
Substitute:

- 1 for $|$,
- $z^{m-1}$ for  and , and
- $z^{m+1}$ for  and 

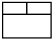
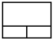
$$T_m(z) = \left(1 + z^{m-1} + z^{m+1}\right)\left(1 - (z^{m-1} + z^{m+1})(z^{m-1} + z^{m+1})\right)^{-1}\left(1 + z^{m-1} + z^{m+1}\right)$$

# All height $m$ tilings (odd)

$$T_m = A + B - |$$



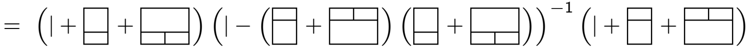
Substitute:

- 1 for $|$,
- $z^{m-1}$ for  and , and
- $z^{m+1}$ for  and 

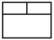
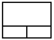
$$\begin{aligned} T_m(z) &= \left(1 + z^{m-1} + z^{m+1}\right)\left(1 - (z^{m-1} + z^{m+1})(z^{m-1} + z^{m+1})\right)^{-1}\left(1 + z^{m-1} + z^{m+1}\right) \\ &= \frac{(1 + z^{m-1} + z^{m+1})^2}{1 - (z^{m-1} + z^{m+1})^2} \end{aligned}$$

# All height $m$ tilings (odd)

$$T_m = A + B - |$$



Substitute:

- 1 for $|$,
- $z^{m-1}$ for  and , and
- $z^{m+1}$ for  and 

$$\begin{aligned} T_m(z) &= \left(1 + z^{m-1} + z^{m+1}\right)\left(1 - (z^{m-1} + z^{m+1})(z^{m-1} + z^{m+1})\right)^{-1}\left(1 + z^{m-1} + z^{m+1}\right) \\ &= \frac{(1 + z^{m-1} + z^{m+1})^2}{1 - (z^{m-1} + z^{m+1})^2} \\ &= \frac{1 + z^{m-1} + z^{m+1}}{1 - z^{m-1} - z^{m+1}} \end{aligned}$$

# The full (ordinary) generating function for the number of tatami tilings of height $m$

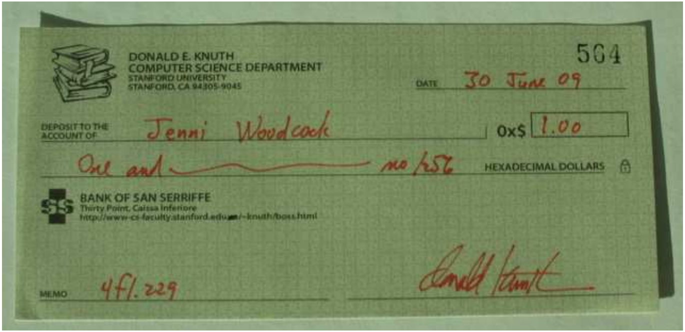
$$T_m(z) = \sum_{n \geq 0} T(m,n) z^n = \begin{cases} 1 & \text{for } m = 0 \\ \frac{1}{1-z^2} & \text{for } m = 1 \\ \frac{1+z^2}{1-z-z^3} & \text{for } m = 2 \\ \frac{1+z^{m-1}+z^{m+1}}{1-z^{m-1}-z^{m+1}} & \text{for } m \text{ odd, } 3 \leq m \leq n \\ \frac{(1+z)(1+z^{m-2}+z^m)}{1-z^{m-1}-z^{m+1}} & \text{for } m \text{ even, } 4 \leq m \leq n, \end{cases}$$

# Results

6 CHANGES TO V4F1: BITWISE TRICKS/TECHNIQUES; BDDS

**Page 229** last lines of answer 215 ——— 10 Jun 2009

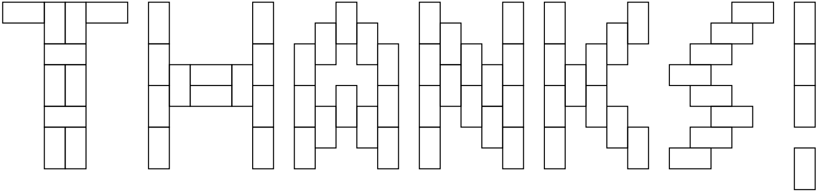
413–420.] ...$(1 - z^{m-1} - z^{m+1})$. ⤳ 413–420. The set of all tatami tilings has been characterized by Dean Hickerson; the corresponding generating functions have been obtained by Frank Ruskey and Jennifer Woodcock (to appear).]



# What we're working on now

Introducing the monomer: □

- odd by odd case with one monomer
- structural characteristics around monomers
- generating functions for different numbers of monomers
- generating functions for different sizes of rectangles
- patterns of growth




Any Questions?