UNIVERSITY OF VICTORIA

APRIL EXAMINATIONS 2002

MATHEMATICS 422 [S01]

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TO BE ANSWERED IN BOOKLETS
DURATION: 3 hours
80 MARKS AVAILABLE

STUDENTS MUST COUNT THE NUMBER OF PAPERS IN THIS EXAMINATION PAPER BEFORE BEGINNING TO WRITE, AND REPORT ANY DISCREPANCY IMMEDIATELY TO THE INVIGILATOR.

THIS QUESTION PAPER HAS 1 PAGE, PLUS COVER.

1. (a) [5] Evaluate

$$\sum_{k=0}^{n} k^2 \binom{n}{k}.$$

(b) [5] Prove the identity in (a) using a combinatorial argument.

2. (a) [5] Prove that the number of partitions of n into odd parts equals the number of partitons of n into distinct parts.

(b) [5] Prove that the number of partitions of n into odd parts equals the number of self-conjugate partitions of n.

3. [10] Prove that the (signless) Stirling number of the first kind, $\begin{bmatrix} n \\ k \end{bmatrix}$, equals the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k cycles. Include formal statements of any theorems used.

4. Let B_n be the n-th Bell number.

(a) [2] Prove that $B_n = \left\{ {n \atop 1} \right\} + \left\{ {n \atop 2} \right\} + \cdots + \left\{ {n \atop n} \right\}$.

(b) [8] Prove that the exponential generating function for the Bell numbers is $e^{e^x - 1}$. Clearly identify any theorems that you use.

5. [10] Use a combinatorial argument (with inclusion-exclusion) to prove

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} \binom{n-k}{m-k} = 0.$$

6. [8] A computer dating service wants to match 4 women each with one of 5 men. Woman 1 is incompatible with men 3 and 5. Woman 2 is incompatible with men 1 and 2. Woman 3 is incompatible with man 4. Woman 4 is incompatible with men 2 and 4. How many matches of the four women are there?

7. (a) [5] Prove that the Pigeonhole Principle is equivalent to the statement

$$R(k_1, k_2, \ldots, k_t; 1) = k_1 + k_2 + \cdots + k_t - t + 1.$$

(b) [5] Prove directly (i.e., without appealing to a more general result) that if $a \geq 2$ and $b \geq 2$ are integers, then $R(a, b)$ is finite and

$$R(a, b) \leq R(a-1, b) + R(a, b-1).$$

(c) [2] Prove that $R(3, 4) = 9$. State any results used.

8. (a) [5] Given a large supply of beads of 5 different colours, how many different necklaces of 9 beads can be made? Two necklaces are the same if one can be rotated and/or flipped to obtain the other.

(b) [5] How many necklaces in (a) use 4 red beads, 3 white beads and 2 black beads?

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