UNIVERSITY OF VICTORIA

APRIL EXAMINATIONS 2003

MATHEMATICS 422 [S01]

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TO BE ANSWERED IN BOOKLETS
DURATION: 3 hours
75 MARKS AVAILABLE

STUDENTS MUST COUNT THE NUMBER OF PAPERS IN THIS EXAMINATION PAPER BEFORE BEGINNING TO WRITE, AND REPORT ANY DISCREPANCY IMMEDIATELY TO THE INVIGILATOR.

THIS QUESTION PAPER HAS 3 PAGES, PLUS COVER.

PART A. DO ALL QUESTIONS

A1. [5] Evaluate

$$\sum_{k=0}^{n} \frac{1}{k+2}\binom{n}{k}.$$

A2. [8] Prove that the (signless) Stirling number of the first kind, $\left[\begin{matrix} n \\ k \end{matrix}\right]$, equals the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k cycles. Include formal statements of any theorems used.

A3. Let $a(n, k)$ be the number of k-permutations of n distinct objects (that is, the number of linear arrangements of k of the n objects), and define $a(n, k)$ to be zero if $k > n$.

(a) [3] Find a recurrence relation and initial conditions for $a(n, k)$.

(b) [6] Let $G_n(x) = \sum_{k=0}^{\infty} a(n, k)\frac{x^k}{k!}$. Find a closed form for $G_n(x)$ and use it to determine $a(n, k)$.

A4. [5] State both the Pigeonhole Principle and Ramsey's Theorem, and explain how Ramsey's Theorem is a generalisation of the Pigeonhole Principle.

A5. [8] Two n-digit sequences consisting of digits 0, 1, 6, 8, 9 are *vertically equivalent* if reading one upside down produces the other. (For example, 0068 and 8900 are vertically equivalent.) How may different (vertically inequivalent) n-digit $(0, 1, 6, 8, 9)$-sequences are there?

PART B. DO ANY FOUR QUESTIONS

B1. A circular merry-go-round with 12 places is to be constructed using horses of three different types: Arabians, Belgians, and Clydesdales. The merry-go-round rotates clockwise, but the horses may be installed facing either direction.

(a) [3] How many different merry-go-rounds can be constructed?

(b) [7] How many of these use four Arabians, two Belgians, and six Clydesdales?

B2 (a) [4] Give a combinatorial argument to show that

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

(b) [6] Prove that

$$\binom{n+m}{m} = \sum_{k=1}^{m} (-1)^{k-1} \binom{n+m}{k} \binom{n+m-k}{n}.$$

(Hint: As a first step, what would happen on the RHS if $k > m$?)

B3. [10] The Math-Stat Course Union is selling solved (old) exams for \$1 each. Suppose that over the course of a day c Calculus exams and d Discrete Math exams are sold. If the cashier starts with no change, one exam is sold at a time, and discrete math students use only \$2 coins, how many sequences of sales allow change to be made whenever it is needed? Assume Calculus exams are always purchased with exact change, and Discrete Math exams are only purchased by discrete math students, who never purchase Calculus exams.

B4 (a) [7] Derive a recurrence relation and initial conditions for s_n, the number of solutions to

$$x_1 + x_2 + \cdots + x_{2n} = 0$$

such that

$$x_i \in \{-1, 1\}, \text{ and}$$
$$x_1 + x_2 + \cdots + x_k \geq 0, \quad 1 \leq k \leq 2n$$

(b) [3] Give a closed form formula (not a recurrence) for s_n. State any results used.

B5 (a) [5] For $0 \leq k \leq n$, find a formula that does not involve a double summation for the number of permutations $x_1 x_2 \ldots x_n$ of $\{1, 2, \ldots, n\}$ such that $x_i = i$ for at least k values of i.

(b) [5] Use rook polynomials to obtain a formula for the number of derangements of $\{1, 2, \ldots, n\}$.

B6 (a) [7] Prove that if $R(a-1, b)$ and $R(a, b-1)$ are both even and greater than two, then

$$R(a, b) \leq R(a-1, b) + R(a, b-1) - 1.$$

(b) Prove that $R(3, 4) = 9$.

B7. [10] This question involves the number of ways to place n balls into m boxes under various conditions. In each case, give a formula for the answer and a short explanation of your reasoning (a proof is not required). Note that your answer may need to take several cases into account.

(a) The balls are distinguishable and the boxes are distinguishable.

(b) The balls are distinguishable and the boxes are identical.

(c) The balls are identical and the the boxes are identical.

(d) The balls are identical, the boxes are distinguishable, and no box can be left empty.

(e) The balls are distinguishable, the boxes are distinguishable, and no box can be left empty.

(f) The balls are distinguishable, the boxes are distinguishable, and box i contains exactly k_i balls, $1 \leq i \leq m$.

(g) The balls are distinguishable, the boxes are identical, and no box can be left empty.

(h) The balls are identical, the boxes are identical, and no box can be left empty.

(i) The balls are identical, the boxes are identical, and box i contains exactly k_i balls, $1 \leq i \leq m$.

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