Math 422/522 Midterm NAME: ______________________________

1. [5 marks] How does $n!$ compare with B_n ? I.e., is is always larger, always smaller, neither? Recall that B_n is the n-th Bell number, the total number of partitions of an n-set. Hint: Compare $\left\{ {n \atop k} \right\}$ and $\left[{n \atop k} \right]$.

2. [5 marks] By comparing coefficients in *exponential* generating functions, what identity is implied by the equation $e^{1+z} = e \cdot e^z$?

3. [5 marks] Recall that $p(n)$ is the number of partitions of an integer n. What generating function equation arises by classifying partitions according to d, the size of their Durfee squares? Fill in the right-hand side of the sum below.

$$P(z) = \sum_{n \geq 1} p(n) z^n = \sum_{d \geq 1}$$

Hint: In class we showed that $\sum_{d \geq 1} z^{d^2} / ((1-z^2) \cdots (1-z^{2d}))$ was the generating function for the number of self-conjugate partitions.

4. [5 marks] How many ways are there to place n labelled balls into k labelled *tubes*? A tube is like a box except that the order in which the balls are placed into the tubes matters; each tube contains a sequence of balls, not a set of balls. HINT: Given that the first $n-1$ balls have been placed, how many places are there for the n-th ball?

5. [5 marks] An inversion in a permutation $\pi_1, \pi_2, \ldots, \pi_n$ is an out-of-order pair; i.e., a pair $i < j$ for which $\pi_i > \pi_j$. For example, the identity permutation has zero inversions and the reversal of the identity has $\binom{n}{2}$ inversions. Give a (very) simple sign-reversing involution that shows that the number of permutations with an even number of inversions is equal to the number with an odd number of inversions.

6. [5 marks] The coefficient of z^n in the generating function below is the solution to a counting problem. What is that counting problem?

$$A(z) = \frac{(1+z)(1+z^3)(1+z^5)\cdots(1+z^{99})}{(1-z^2)(1-z^4)\cdots(1-z^{100})}.$$

Answer the same question for

$$B(z) = \frac{(1+2z)^{100}}{1-z^{100}}.$$

7. [5 marks] What is the functional equation describing the generating function of binary trees with n nodes? Of extended binary trees with n leaves? Of unordered extended binary trees with n leaves? Just give the equation, do not solve it. Hint: binary tree = empty or left and right subtree; extended binary tree = leaf or left and right subtree; unordered extended binary tree = leaf or two unordered subtrees. Use y for the generating function and z for the variable.