Math 422/522 Midterm NAME: I.M. Solution

1. [5 marks] How does $n!$ compare with B_n ? I.e., is is always larger, always smaller, neither? Recall that B_n is the n-th Bell number, the total number of partitions of an n-set. Hint: Compare $\left\{ {n \atop k} \right\}$ and $\left[{n \atop k} \right]$.

   ANSWER: By ordering the elements of the blocks of a set partition we obtain the cycles of a permutation. Thus it is always the case that $B_n \le n!$.

2. [5 marks] By comparing coefficients in *exponential* generating functions, what identity is implied by the equation $e^{1+z} = e \cdot e^z$?

   ANSWER: The equation is equivalent to $e^{(1+z)x} = e^x \cdot e^{zx}$. Now convolute:

$$e^{(1+z)x} = \sum_{n\ge 0}(1+z)^n \frac{x^n}{n!} = e^x \cdot e^{zx} = \sum_{n\ge 0} \frac{x^n}{n!} \sum_k \binom{n}{k} z^k.$$

   Thus the identity implied is the binomial formula $(1+z)^n = \sum_k \binom{n}{k} z^k$.

3. [5 marks] Recall that $p(n)$ is the number of partitions of an integer n. What generating function equation arises by classifying partitions according to d, the size of their Durfee squares? Fill in the right-hand side of the sum below. Hint: In class we showed that $\sum_{d\ge 1} z^{d^2}/((1-z^2)\cdots(1-z^{2d}))$ was the generating function for the number of self-conjugate partitions.

   ANSWER:

$$P(z) = \sum_{n\ge 1} p(n) z^n = \sum_{d\ge 1} \frac{z^{d^2}}{(1-z)^2(1-z^2)^2\cdots(1-z^d)^2}$$

4. [5 marks] How many ways are there to place n labelled balls into k labelled *tubes*? A tube is like a box except that the order in which the balls are placed into the tubes matters; each tube contains a sequence of balls, not a set of balls. HINT: Given that the first $n-1$ balls have been placed, how many places are there for the n-th ball?

   ANSWER:

$$k(k+1)\cdots(k+n-1) = \frac{(n+k-1)!}{(k-1)!} = n!\binom{n+k-1}{n}.$$

5. [5 marks] An inversion in a permutation $\pi_1, \pi_2, \ldots, \pi_n$ is an out-of-order pair; i.e., a pair $i < j$ for which $\pi_i > \pi_j$. For example, the identity permutation has zero inversions and the reversal of the identity has $\binom{n}{2}$ inversions. Give a (very) simple sign-reversing involution that shows that the number of permutations with an even number of inversions is equal to the number with an odd number of inversions.

   ANSWER: Swap the first two elements.

6. [5 marks] The coefficient of z^n in the generating function below is the solution to a counting problem. What is that counting problem?

$$A(z) = \frac{(1+z)(1+z^3)(1+z^5)\cdots(1+z^{99})}{(1-z^2)(1-z^4)\cdots(1-z^{100})}.$$

ANSWER: Number of partitions of an integer n where each part is at most 100 and any odd parts are distinct.

Answer the same question for

$$B(z) = \frac{(1+2z)^{100}}{1-z^{100}}.$$

ANSWER: Imagine a supply of 100 labelled pennies and an unlimited supply of (unlabelled) loonies. The coefficient of z^n in the generating function is the sum of 2^p taken over all ways of making change for the amount n using those coins, where p is the number of pennies used. (Yes, this is a rather convoluted counting problem!)

7. [5 marks] What is the functional equation describing the generating function of binary trees with n nodes? Of extended binary trees with n leaves? Of unordered extended binary trees with n leaves? Just give the equation, do not solve it. Hint: binary tree = empty or left and right subtree; extended binary tree = leaf or left and right subtree; unordered extended binary tree = leaf or two unordered subtrees. Use y for the generating function and z for the variable.

ANSWER:
(a) $y = 1 + zy^2$,
(b) $y = z + y^2$,
(c) $y(z) = z + (y(z)^2 + y(z^2))/2$.

The answer for part (c) is tricky; it comes from the recurrence relation

$$a_n = \sum_{\substack{i+j=n \\ i<j}} a_i a_j + [\![n \text{ even}]\!] \binom{a_{n/2}+1}{2}.$$

We will encounter this again when we count trees in the final chapter.