Homework #1.

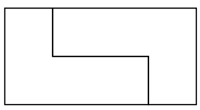
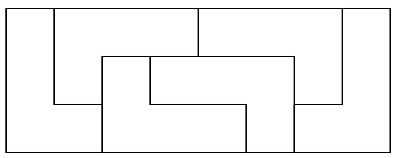
- 1.4. (b) Note that there are $2n-1$ of the 45 degree diagonals; at most one bishop can lie on any of these diagonals. Furthermore, there are two diagonals of length 1 and these cannot both contain bishops since they both lie on the same -45 degree diagonal. Thus the maximum number is at most $2n-2$. This value many be achieved for example by placing $n-1$ bishops in the leftmost column starting at the top, and similarly for the rightmost column.

  (c) Colour the squares of the chessboard white and black in the usual way. Note that bishops only attack squares of the same colour. Thus there are two independent problems: place $n-1$ non-taking bishops on the black squares, and place $n-1$ non-taking bishops on white squares. Let the number of solutions to these problems be B and W, respectively. The total number of solutions is BW. If n is even, then the structure of the white diagonals is the same as the structure of the black diagonals and thus $B = W$ from which it follows that BW is a square.

- 1.11. (a) The graph has 12 vertices of degree 3 and 9 vertices of degree 4. Each degree 4 vertex is adjacent only to vertices of degree 3 and vice-versa; the graph is therefore bipartite. Alternating between the two partite sets, the maximum length path would have to contain 10 degree 3 vertices and 9 degree 4 vertices.

- 1.18.

  - (a) Implied by (b) below.
  - (b) Since the L-shape covers 4 squares mn must be even and so we can assume that the number of columns is even. Now colour the columns alternately blue and red. Each L-shape covers 3 of one colour and 1 of the other. Let x be the number of L-shapes covering 3 red and 1 blue, and let y be the number covering 1 red and 3 blue. Since there are equal numbers of red and blue squares, $3x + y = 3y + x$, and thus $x = y$. For the total number of L-shapes, $x + y = 2x = mn/4$. It follows that 8 divides mn.

    A  B 

  - (c) There are two cases to consider; (c1) $2 \mid m$ and $4 \mid n$ and (c2) m odd and $8 \mid n$. Note from the figure below that we can construct solutions A for 2×4 and B for 3×8. In case (c1) we can use a $m/2 \times n/4$ grid of the A configuration. In case (c2) we can write $m = 3 + 2s$ where s is a natural number, and then place a row of s of the B configurations atop a $s \times n/4$ grid of the A configurations.

- 1.25. Call the 4 conditions (a), (b), (c), (d). If $1 \in A$, then $1 + 1 = 2 \in A$, and so on; every number is in A, which contradicts $B \neq \emptyset$. Thus $1 \in B$. Since $1 \in B$, by (c),

$1+1=2\in A$. Applying (b) repeatedly, every even number is in A. Aside from 1, every other odd number can be written as $x=y+1$ where $y\geq 2$. Thus, by rule (d), $x\in B$. [Note: its seems like the condition $A\neq\emptyset$ is not used.]

- 2.5. There are $\binom{2n}{n}$ ways to select n elements from the set $X\cup Y=\{x_1,x_2,\ldots,x_n\}\cup\{y_1,y_2,\ldots,y_n\}$. Let S be such a selection. Consider the involution that takes the smallest j such that exactly one of x_j or y_j is in S and replaces the one that is not in with the one that is in S. There are two fixed points, namely when $S=X$ or $S=Y$. An involution with an even number of fixed points can only be defined on a set with an even number of elements.

- 2.15. Use the shepherd principle. The number of m-permutations of a n-set is $(n)_m=n!/(n-m)!$. Each equivalence class under rotation contains the same number of elements, namely m.

- 2.47. Let $n_1<n_2<\cdots<n_t$ be the positions of ones in the binary expansion of n, so that

  $$n=\sum 2^{n_i}.$$

  Note that mod 2, we can (inductively) write $(1+x)^{2^m}$ as

  $$(1+x)^{2^{m-1}}(1+x)^{2^{m-1}}=(1+x^{2^{m-1}})(1+x^{2^{m-1}})=1+2x^{2^{m-1}}+x^{2^m}=1+x^{2^m}.$$

  Computing mod 2,

  $$(1+x)^n=\prod_{i=1}^{t}(1+x)^{2^{n_i}}=\prod_{i=1}^{t}1+x^{2^{n_i}}=\sum_{i=1}^{t}\alpha_i x^i,$$

  where α_i is $\binom{n}{i}$ mod 2. The second product clearly expands into 2^t terms.

- 2.55. We are asked to prove that

  $$\sum_k\binom{n-k}{m-k}\binom{r+k}{k}=\binom{n+r+1}{m}$$

  Following the suggestion let k be the number of the dot, counting from 0 and starting at the bottom. There are $\binom{r+k}{k}$ paths that start at the origin and pass through the k-th dot. There are $\binom{n-k}{m-k}$ paths starting at the k-th dot and ending at $(n-m+r+1,m)$.

- 2.62 (a). We want to prove that

  $$\frac{n!}{x(x+1)\cdots(x+n)}=\sum_{k=0}^{n}\frac{(-1)^k}{x+k}\binom{n}{k}.$$

Consider the partial fraction expansion

$$\frac{n!}{x(x+1)\cdots(x+n)} = \sum_{k=0}^{n} \frac{\alpha_k}{x+k}.$$

Fix $0 \le r \le n$, multiply both sides by $x+r$, and then let $x = -r$. We get

$$\alpha_r = \frac{n!}{-r(-r+1)\cdots(-r+r-1)(-r+r+1)\cdots(-r+n)} = \frac{(-1)^r n!}{r!(n-r)!}.$$

- 62 (b). Plugging in $x = 1$ we get

  $$\frac{1}{n+1} = \sum_{k\ge 0} \frac{(-1)^k}{k+1}\binom{n}{k}$$

  By the binomial expansion of $(1-1)^{n+1}$,

  $$0 = \sum_{k\ge 0}\binom{n+1}{k}(-1)^k = 1 + \sum_{k\ge 1}\frac{n+1}{k}\binom{n}{k-1}(-1)^k = 1 - \sum_{k\ge 0}\frac{n+1}{k+1}\binom{n}{k}(-1)^k.$$

  Now bring the sum to the other side of the equation and divide by $n+1$.