Homework #1.

- 3.1. This is the binomial expansion

$$1 = ((1+x)+(-x))^n = \sum_k \binom{n}{k}(-x)^k(1+x)^{n-k}.$$

- 3.9. We use below the identity $\binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$, which is valid so long as n is a non-zero integer. Each sum below is summed over integer k.

$$\begin{aligned}
0 &= \left(1-\frac{1}{1+nx}\right)^n - \left(1-\frac{1}{1+nx}\right)^n \\
&= \left(1-\frac{1}{1+nx}\right)^n - \frac{nx}{1+nx}\left(1-\frac{1}{1+nx}\right)^{n-1} \\
&= \sum \binom{n}{k}\frac{(-1)^k}{(1+nx)^k} - \frac{nx}{1+nx}\sum\binom{n-1}{k}\frac{(-1)^k}{(1+nx)^k} \\
&= \sum \binom{n}{k}\frac{(-1)^k}{(1+nx)^k} + \frac{nx}{1+nx}\sum\binom{n-1}{k-1}\frac{(-1)^k}{(1+nx)^{k-1}} \\
&= \sum \binom{n}{k}\frac{(-1)^k}{(1+nx)^k} + \frac{nx}{1+nx}\sum\frac{k}{n}\binom{n}{k}\frac{(-1)^k}{(1+nx)^{k-1}} \\
&= \sum \binom{n}{k}\frac{(-1)^k}{(1+nx)^k} + x\sum k\binom{n}{k}\frac{(-1)^k}{(1+nx)^k} \\
&= \sum \binom{n}{k}\frac{(1+kx)(-1)^k}{(1+nx)^k}.
\end{aligned}$$

For part (d) simply halve the number of occurrences of each element selected.

- 3.10. This one is quite straight forward.

$$\left(\sum_{k\geq 0} x^{2k}\right)^n = \left(\frac{1}{1-x^2}\right)^n = \sum_{k\geq 0}\binom{-n}{k}(-x)^{2k} = \sum_{k\geq 0}\binom{n+k-1}{k}x^{2k}$$

- 3.41.

(a) Classify the selections according to whether n is selected or not. If it is not selected then we still have to select k element from the remaining n, but circularity does not matter any more so there are $f(n-1,k)$ ways to select. If n is selected then $n-1$ and 1 cannot be selected; we still then need to select $k-1$ elements from the remaining $n-3$; again circularity will not matter so the number of ways is $f(n-3,k-1)$.

(b) Use part (b) of Exercise 3.40 and some algebra.

- 3.23. We proceed by induction on n to prove that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. If $n = 2$ then $F_3F_1 - F_2^2 = 2 - 1 = +1$. Otherwise,

$$\begin{aligned}
(-1)^n &= F_{n+1}F_{n-1} - F_n^2 \\
&= F_{n+1}(F_{n+1} - F_n) - F_n^2 \\
&= F_{n+1}^2 - F_{n+1}F_n - F_n^2 \\
&= F_{n+1}^2 - (F_{n+2} - F_n)F_n - F_n^2 \\
&= F_{n+1}^2 - F_{n+2}F_n + F_n^2 - F_n^2 \\
&= F_{n+1}^2 - F_{n+2}F_n
\end{aligned}$$

- Extra Question. The solution to the question is

$$T(n,k) = \frac{k}{2n-k}\binom{2n-k}{n-k}.$$

Cycle Lemma and Shepherd Principle:
Think of $\mathbf{T}(n,k)$ as the set of bitstrings of the form $1^k0b_{k+2}\cdots b_{2n}$ with n 1's, n 0's and satisfying the property that there are at least as many 1's as 0's in any prefix (the *prefix* property). Observe that the string is such a possible suffix if and only if

$$a_1a_2\cdots a_{2n-k-1} = \bar{b}_{2n}\cdots\bar{b}_{k+2}$$

satisfies that same prefix property. Now consider the string $1a_1\cdots a_{2n-k-1}$. It contains n 1's, $n-k$ 0's and is characterized by the property that every prefix has *more* 1's than 0's (the *strong prefix* property). We count the number of such strings using the cycle lemma. The total number of bitstrings with n 1's and $n-k$ 0's is $\binom{2n-k}{n-k}$. By the cycle lemma we multiply by the fraction $k/(2n-k)$ to get the number of such bitstrings that satisfy the strong prefix property. Note that we can not argue (as was done in class) that each equivalence class under rotations has the same number of elements. For example, if $n = 6$ and $k = 4$ then the equivalence class of $A = 110110$ has 3 distinct strings but the equivalence class of $B = 111100$ has 6 distinct strings. However, there is one valid starting point for A and 2 valid starting points for B, so the fraction is the same in either case.

Reflection Principle:

We can think of $T(n,k)$ as the number of increasing walks on the integer lattice that start at $(0,0)$ end at $(n-1, n-k)$ and that stay below the diagonal thru the points $(-1,0)$ and $(0,1)$. There are $\binom{2n-k-1}{n-k}$ unrestricted increasing walks. The number of walks that intersect the diagonal can be determined by reflecting the them about their last intersection with the diagonal. Such walks start at the origin and terminate at $(n-k-1, n)$. Thus

$$t(n,k) = \binom{2n-k-1}{n-k} - \binom{2n-k-1}{n-k-1} = \frac{k}{2n-k}\binom{2n-k}{n-k}.$$

Generating Functions:

$$\begin{aligned}
T(x,y) &= \sum_{n\geq 0}\sum_{k\geq 0} t(n,k)x^n y^k \\
&= \sum_{n\geq 0}\sum_{k\geq 0}\sum_{\substack{\nu_1+\nu_2+\cdots+\nu_k=n \\ \nu_i\geq 1}} C_{\nu_1}C_{\nu_2}\cdots C_{\nu_k}x^n y^k \\
&= \sum_{k\geq 0} y^k \left(\sum_{\nu\geq 1} C_\nu x^\nu\right)^k \\
&= \sum_{k\geq 0}(y(G(x)-1))^k \\
&= \frac{1}{1-y(G(x)-1)}
\end{aligned}$$