Homework #4.

- **5.5.** (a) Every region in B will match n times with a region from A. There are $2n$ regions; thus the total number of matches is $2n^2 = a_1 + a_2 + \cdots + a_{2n} a_1 + a_2 + \cdots + a_{2n}$. (b) If all $a_i < n$ then $a_1 + a_2 + \cdots + a_{2n} \leq (n-1)2n$, a contradiction.

- **5.18.** By $y \mod 1$ denote the quantity $y - \lfloor y \rfloor$. Clearly $0 \leq y \mod 1 < 1$. Let $X = \{1x \mod 1, 2x \mod 1, \ldots (n-1)x \mod 1\}$. Divide the range $[0, 1)$ into n equal sized segments. If some number in X falls into the first or last segment then we are done. Otherwise let the remaining $n-2$ segments be the pigeon holes. Two of the $n-1$ pigeons in X must lie in the same segment, say ix and jx where $i < j$. Then $|ix \mod 1 - jx \mod 1| \leq 1/n$. Hence $|(j-i)x \mod 1| \leq 1/n$, which imples that $(j-i)x$ is within $1/n$ of an integer.

- **5.21** By Theorem 57 and the Pascal triange recurrence relation, the result will follow by induction so long as the initial conditions are satisfied. Clearly if $a = 1$ or $b = 1$ then $N(a, b) = 1$ (the 1-clique has 0 edges).

- **5.22** Number the vertices $0, 1, 2, 3, 4, 5, 6, 7$. Let the red edges be $\{(i, i+1), (i, i+4) : 0 \leq i < 8\}$, with addition mod 8, and color all of the others blue. There are no red triangles in this graph (the smallest red cycle has length 5). The graph is vertex transitive. Vertex 0 is incident via blue edges with the vertices 2,3,5,6. A blue 4-clique must use at least 3 of those, but such a selection would include either 2,3 or 5,6, which are coloured red, a contradiction.

- **5.47** Let P_j be the property that person j faces the same person they faced before. If k of the properties are specified then there remains $(n-k-1)!$ ways to arrange the remaining $n-k$ persons in a circle. Thus (with $(-1)! = 1$)

  $$a_n = \sum_{k \geq 0} \binom{n}{k} (-1)^k (n-k-1)!.$$

  We now show that $D_n = a_n + a_{n+1}$, which will prove the result.

  $$\begin{aligned} a_n + a_{n+1} &= \sum_{k \geq 0} \binom{n}{k} (-1)^k (n-k-1)! + \sum_{k \geq 0} \binom{n+1}{k} (-1)^k (n-k)! \\ &= -\sum_{k \geq 0} \binom{n}{k-1} (-1)^k (n-k)! + \sum_{k \geq 0} \left[\binom{n}{k} + \binom{n}{k-1} \right] (-1)^k (n-k)! \\ &= \sum_{k \geq 0} \binom{n}{k} (-1)^k (n-k)! \end{aligned}$$

- **5.48** Let P_j be the property that a_j is adjacent to a_j (and thus may be treated as a single symbol). If k of the properties hold, then there are $2n-k$ symbols and $n-k$ of them come in pairs.

$$\sum_{k\geq 0}\binom{n}{k}(-1)^k\frac{(2n-k)!}{2^{n-k}}.$$

- **5.56.**

$$\begin{aligned}
n &= \sum_{k=1}^{n} 1 \\
&= \sum_{k=1}^{n}\sum_{d|n} [\![\gcd(n,k)=d]\!] \\
&= \sum_{d|n}\sum_{k=1}^{n} [\![\gcd(n,k)=d]\!] \\
&= \sum_{d|n}\sum_{k=1}^{n} [\![\gcd(n/d,k/d)=1]\!] \\
&= \sum_{d|n}\sum_{k=1}^{n/d} [\![\gcd(n/d,k)=1]\!] \\
&= \sum_{d|n}\phi(n/d) \;=\; \sum_{d|n}\phi(d)
\end{aligned}$$