Homework #5.

- **6.10.** (a) Let $\pi = (1\ 2\ \cdots\ n)$. The length of the cycle containing i in π^k is the smallest value j such that

  $$\overbrace{\pi^k(\pi^k(\cdots\pi^k}^{j}(i)\cdots)) = \pi^{jk}(i) = i.$$

  Since $\pi(i) = i+1 \bmod n$, we have $i+jk = i \bmod n$, and hence $jk = 0 \bmod n$. In other words, j is the smallest non-zero number for which $jk = 0 \bmod n$. This quantity is independent of i and thus all cycles have the same length.

  (b) Let j be the common cycle size in a regular permutation ρ, and let $p = n/j$. Define a_{cp} to be the element in cycle c with position p. Define a circular permutation π whose elements are

  $$\pi = (a_{11}, a_{21}, \cdots, a_{j1}, \ldots, a_{1p}, a_{2p}, \ldots, a_{jp}).$$

  Clearly, $\pi^j = \rho$.

  (c) By the Cauchy formula, Theorem 72,

  $$\sum_{d|n} \frac{n!}{d!(n/d)^d}.$$

  (d) Let $\rho = (1\ 2\ \cdots\ n)$. Clearly, any permutation commutes with powers of itself. Now suppose that $\rho\pi = \pi\rho$ for some $\pi \in S_n$. For $x \in \{1, 2, \ldots, n\}$, note that $x\rho\pi = (x+1)\pi$ and $x\pi\rho = x\pi+1$, with addition (here and below) done mod n. Thus $(x+1)\pi = x\pi+1$ and hence π is determined by the value of $y = 1\pi$ in the sense that we must have

  $$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ y & y{+}1 & \cdots & y{+}n{-}1 \end{pmatrix}.$$

  This proves what we want, namely $\pi = \rho^{y-1}$.

- **6.40.** The parity of a j-cycle is equal to the parity of $j-1$. The parity of the product of cycles is equal to the product of their parities (whether they are disjoint or not). Thus the parity of a permutation of specification $1^{a_1}2^{a_2}\cdots n^{a_n}$ is

  $$\sum_{j=1}^{n}(j-1)a_j = \sum_{j \text{ even}} a_j \pmod 2.$$

- **6.46.** Note that $a_1 + \cdots + a_n$ is the number of disjoint cycles in a permutation of specification $1^{a_1}2^{a_2}\cdots n^{a_n}$. Thus $n!$ times the sum is equal to the number of permutations in S_n with an even number of disjoint cycles minus the number with an odd number of disjoint cycles. Denote by $\rho(\pi)$ the number of cycles in permutation π. Consider the multiplication of a permutation π by the transposition $\tau = (x\ y)$. If x and y are in the same cycle c in π, then in $\tau\pi$ the cycle c is split into two cycles, one containing x and the other containing y (and the rest of the permutation unchanged), and thus $\rho(\tau\pi) = \rho(\pi) + 1$. Conversely, if they are in different cycles in π, then in $\tau\pi$ those two cycles are coalesced into a single cycle containing both x and y (and the rest of the permutation unchanged), and thus $\rho(\tau\pi) = \rho(\pi) - 1$. Thus multiplication by τ changes the number of cycles ± 1 and is therefore a sign-reversing involution without fixed points. Therefore the signed sum is zero.

- **6.47.** Recall again that $a_1 + \cdots + a_n$ is the number of disjoint cycles in a permutation. Thus, multiplying by $n!$, the sum must be the ordinary generating function of the number of permutations with a given number of cycles. I.e.,

$$\begin{aligned}
\sum_{\pi \in S_n} x^{\rho(\pi)} &= \sum_{a_1+2a_2+\cdots+na_n=n} \frac{n!}{a_1!a_2!\cdots a_n! 1^{a_1} 2^{a_2} \cdots n^{a_n}} x^{a_1+a_2+\cdots+a_n} \\
&= \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \\
&= (-1)^n \sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} (-x)^k \\
&= (-1)^n (-x)(-x-1)\cdots(-x-n+1) \\
&= x(x+1)\cdots(x+n-1) \\
&= n! \binom{x+n-1}{n} \\
&= \langle t^n \rangle \frac{1}{(1-t)^x}
\end{aligned}$$

The fourth equality is from the definition of the $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ numbers on page 118 of the text.

- **6.58.** (a) Imagine the cube sitting on a table. There are 6 choices for the top face. Once the top face is fixed there are 4 possible choices for the face that faces you. Once that face is fixed, the entire cube is fixed. Thus there are $4 \cdot 6 = 24$ orientations.

  (b) We classify the 24 group elements according to the possible axes of rotation. An axis of rotation passes through either opposite edges, opposite faces, or opposite vertices.

identity	edges	faces	vertices
(1)(2)(3)(4)(5)(6)	(16)(24)(35)	(1624)	(163)(245)
	(14)(26)(35)	(1426)	(136)(254)
	(13)(25)(46)	(3456)	(134)(256)
	(15)(23)(46)	(3654)	(143)(265)
	(12)(34)(56)	(1325)	(156)(234)
	(12)(36)(45)	(1523)	(165)(243)
		(12)(46)	(154)(236)
		(35)(46)	(145)(263)
		(12)(35)	

- **6.59.** This can be solved by the shepherd principle, but we do it with Burnside's lemma. The group G under consideration consists of permutations of all 6! colorings of the cube by 6 distinct colours. There are 24 such permutations by problem 58. However, note that none of them except the identity leave any coloring fixed. Thus $\lambda_1(g) = 6!\llbracket g = I \rrbracket$ and so by Burnside's Lemma the number of orbits (distinct colourings) is

$$\frac{1}{24}\sum_{g\in G}\lambda_1(g) = \frac{6!}{24} = 30.$$

- **6.60.** The group elements from question 58 have the following cycle structures and frequencies of occurence and number of cycles:

g	freq.	$\lambda(g)$
1^6	1	6
2^3	6	3
$1^2 4^1$	6	3
$1^2 2^2$	3	4
3^2	8	2

By Polya's theorem we want to expand

$$\frac{1}{24}\left[(x+y)^6 + 6(x^3+y^3)^2 + 6(x+y)^2(x^4+y^4) + 3(x+y)^2(x^2+y^2)^2 + 8(x^3+y^3)^2\right]$$

which is then equal to

$$G(x,y) = x^6 + x^5y + 2x^4y^2 + 2x^3y^3 + 2x^2y^4 + xy^5 + y^6.$$

The answer to (a) is $2 = \langle x^3y^3\rangle G(x,y)$, the answer to (b) is $2 = \langle x^2y^4\rangle G(x,y)$.

(d) By Theorem 78, the number of such colourings is

$$\frac{1}{|G|}\sum_{g\in G} x^{\lambda(g)} = \frac{1}{24}[x^6 + 3x^4 + 12x^3 + 8x^2]$$

(c) Plugging in $x = 2$ into (d) gives the answer 10.