UNIVERSITY OF VICTORIA
EXAMINATIONS APRIL 2005
MATHEMATICS 422/522 (S01)

NAME: I.M. Solution — STUDENT NO. 0123456
INSTRUCTOR: Frank Ruskey — SECTION: S01
. — DURATION: 3 Hours

TO BE ANSWERED ON THE PAPER

STUDENTS MUST COUNT THE NUMBER OF PAGES IN THIS EXAMINATION PAPER BEFORE BEGINNING TO WRITE, AND REPORT ANY DISCREPANCY IMMEDIATELY TO THE INVIGILATOR.

THIS QUESTION PAPER HAS 10 PAGES.

NOTES: (0) CLOSED BOOK EXAM; NO NOTES OR CALCULATORS ALLOWED, (1) ANSWER ALL QUESTIONS, (2) THERE ARE A TOTAL OF 73 MARKS, (3) SCRATCH PAPER IS AVAILABLE FROM THE INVIGILATORS, BUT NOTE THAT THERE IS BLANK PAGE AT THE END.

Question	Possible marks	Actual marks
1	10	
2	6	
3	5	
4	5	
5	12	
6	10	
7	6	
8	4	
9	5	
10	10	
Total	73	

1. Match the items on the right with the most closely corresponding item on the left by putting a letter in the "answer" column. [10 marks]

	left item	**answer**	**right item**
A	Lagrange inversion	D	binomial expansion
B	de Bruijn sequence	J	$a(n,m) \leq a(n-1,m) + a(n,m-1)$
C	necklace	E	$\binom{n+k-1}{k}$
D	$(1+z)^n$	I	$a(n,m) = a(n-1,m-1) + ma(n,m-1)$
E	$1/(1-z)^n$	B	00010111
F	$e^{-z}/(1-z)$	A	$y = x(1+y^3)$
G	$((1-z)(1-z^5)(1-z^{10})(1-z^{25}))^{-1}$	C	0000,0001,0011,0101,0111,1111
H	$(1-\sqrt{1-4z})/(2z)$	H	ordered forests
I	Stirling number	F	derangements
J	Ramsey number	G	making change

2. Let $P(n)$ be the set of all ways of arranging the integers $1, 2, \ldots, 2n$ subject to the constraint that $2i-1$ is to the left of $2i$ in the arrangement. (a) How many such arrangements are there? (b) Give a sign-reversing involution that shows that the number of arrangements with an even number of inversions minus the number with an odd number of inversions is $n!$. [6 marks]

ANSWER:

(a) The constraint is equivalent to making $2i-1$ and $2i$ into the same symbol; thus we have n pairs of identical symbols from which to make the arrangement. Thus

$$|P(n)| = \binom{2n}{2,2,\ldots,2} = \frac{(2n)!}{2^n} = n!(2n-1)(2n-3)\cdots 3\cdot 1.$$

(b) Let ϕ be the involution that, given an arrangement $\pi_1\pi_2\cdots\pi_{2n}$, reverses the first pair $\pi_{2i-1}\pi_{2i}$ such that $\{\pi_{2i-1},\pi_{2i}\} \neq \{2j-1,2j\}$ for any $j = 1, 2, \ldots, n$. A fixed point occurs if there is no such pair. The fixed points all have the form $2x_1-1, 2x_1, 2x_2-1, 2x_2, \ldots, 2x_n-1, 2x_n$, for some permutation $x_1, x_2, \ldots, x_n$ of $1, 2, \ldots, n$. Thus there are $n!$ of them. We must show that they all have even parity. Observe that $cdab$ has 4 inversions (as compared with $abcd$). Since any fixed point can be transformed into any other by a series of such moves, they all have even parity.

3. Let $A(n)$ denote the set of sequences of non-negative integers $a(1), a(2), \ldots, a(n)$ with the property that $a(j) < j$ and $a(j - a(j)) = 0$ for $j = 1, 2, \ldots, n$. For example, $A(3) = \{000, 001, 002, 010, 012\}$. Show that $\left\{ {n \atop m} \right\}$ is the number of sequences in $A(n)$ where exactly m values $1 \le j \le n$ satisfy $a(j) = 0$. [5 marks]

   ANSWER:

   Proof #1: Let $j_1, j_2, \cdots, j_m$ be the positions for which $a(j) = 0$. Now define the i-th block of a partition $B_1, B_2, \cdots, B_m$ of $\{1, 2, \ldots, n\}$ to be the set

   $$B_i = \{k : k - a(k) = j_i\}.$$

   This rule gives a one-to-one correspondence between our sequences and partitions of an n-set into m blocks, which we know have cardinality $\left\{ {n \atop m} \right\}$.

   Proof #2: Let $a(n, m)$ be the number of sequences in $A(n)$ where exactly m values $1 \le j \le n$ satisfy $a(j) = 0$. Consider the value of $a(n)$. If $a(n) = 0$ then there are $a(n-1, m-1)$ choices for $a(1), a(2), \ldots, a(n-1)$. If $a(n) > 0$ then $n - a(n)$ must be one of the m values j for which $a(j) = 0$ (otherwise, $a(n - a(n)) \ne 0$). Thus there are m choices for $a(n)$ and so $m\ a(n-1, m)$ ways this case can happen. Hence

   $$a(n, m) = a(n-1, m-1) + m\ a(n-1, m),$$

   which is the same recurrence relation that the $\left\{ {n \atop m} \right\}$ numbers satisfy. Thus the numbers are the same if they have the same initial conditions, which is easy to check.

4. Use the pigeon-hole principle to prove the following statement. For any n positive integers $\{a_1, a_2, \ldots, a_n\}$, the sum of some of these integers (perhaps one of the numbers itself) is divisible by n. HINT: a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, etc. [5 marks]

   ANSWER: Let $s_k = a_1 + a_2 + \cdots + a_k$. If some s_k is divisible by n (i.e., $s_k \equiv 0 \bmod n$), then we are done. Otherwise, $s_k \bmod n \in \{1, 2, \ldots, n-1\}$. Since there are n numbers $\{s_1, s_2, \ldots, s_n\}$ and $n - 1$ possible values for $s_k \bmod n$, by the pigeon-hole principle some two of them must be congruent mod n, say $s_i \equiv s_j \bmod n$, where $i < j$. But then $s_j - s_i = x_{i+1} + \cdots + x_j \equiv 0 \bmod n$ as required.

5. In this question we consider the number of solutions to the equation

$$x_1 + x_2 + \cdots + x_k = n$$

subject to various constraints. For each set of constraints below, state the number of solutions in as simple a form as possible. Use $p(n,k)$ to denote the number of partitions of n into k parts. Avoid summations if possible for full marks. For a constraint involving i it must hold for $i = 1, 2, \ldots, k$. [12 marks]

constraint	**answer**
$x_i \in \{0,1\}$	$\binom{k}{n}$
$x_i \in \{0,1,\ldots,n\}$	$\binom{k+n-1}{n}$
$x_i \in \{1,2,\ldots,n\}$	$\binom{n-1}{n-k} = \binom{n-1}{k-1}$
$0 \le x_1 \le x_2 \le \cdots \le x_k$	$p(n+k,k)$
$0 < x_1 < x_2 < \cdots < x_k$	$p(n - \binom{k}{2}, k)$
$x_i \ge 0$ and $x_1 + \cdots + x_i < i$	$\frac{k-n}{k+n}\binom{k+n}{k} = \binom{k+n-1}{k-1} - \binom{k+n-1}{k}$

6. (a) Use inclusion-exclusion to derive a formula for a_n the number of permutations of $\{1, 2, \ldots, n\}$ such that j and $j+1$ are never adjacent for $j = 1, 2, \ldots, n-1$. [5 marks] For example, if $n = 3$ then the permutations are 132, 213, 321.

(b) What is the exponential generating function of the numbers $b_n = a_{n+1}$? [5 marks]

ANSWER: This problem is very similar to the derangements problem. (a) Given a permutation, let k be the size of a set of values j for which j and $j+1$ are adjacent. The total number of such permutations is $\binom{n-1}{k}(n-k)!$ (since $j \ne n$). Thus, by inclusion-exclusion,

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n-1}{k} (n-k)!.$$

(b)

$$\begin{aligned}
\sum_{n\ge 0} b_n \frac{z^n}{n!} &= \sum_{n\ge 0}\sum_{k\ge 0} \binom{n}{k} (-1)^k (n+1-k)! \frac{z^n}{n!} \\
&= \left(\sum_{n\ge 0} (-1)^n \frac{z^n}{n!}\right)\left(\sum_{n\ge 0} (n+1)! \frac{z^n}{n!}\right) \\
&= \frac{e^{-z}}{(1-z)^2}
\end{aligned}$$

The last equality follows from:

$$\sum_{n\ge 0} (n+1) z^n = \sum_{n\ge 0} n z^{n-1} = \frac{d}{dz} \sum_{n\ge 0} z^n = \frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2}.$$

7. What binomial coefficient identity arises from the equation shown below. [6 marks]

$$(1-z^2)^n = (1-z)^n(1+z)^n.$$

ANSWER: Expanding the left-hand side:

$$(1-z^2)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}z^{2k} = \sum_{k\geq 0}[\![k \text{ is even}]\!](-1)^{k/2}\binom{n}{k/2}z^k.$$

Expanding the right-hand side:

$$\begin{aligned}(1-z)^n(1+z)^n &= \sum_{k\geq 0}(-1)^k\binom{n}{k}z^k\sum_{k\geq 0}\binom{n}{k}z^k \\ &= \sum_{k\geq 0}\sum_{j\geq 0}(-1)^j\binom{n}{j}\binom{n}{k-j}z^k\end{aligned}$$

Equating coefficients, we obtain

$$\sum_{j\geq 0}(-1)^j\binom{n}{j}\binom{n}{k-j} = [\![k \text{ is even}]\!](-1)^{k/2}\binom{n}{k/2}$$

8. Explain why, for any $n \geq 1$,

$$n^n \sum_{\substack{\lambda_1+2\lambda_2+\cdots+n\lambda_n=n \\ \lambda_i\geq 0}}\left(\prod_{k=1}^{n}\lambda_k!k^{\lambda_k}\right)^{-1} = n!\sum_{\substack{\nu_1+\nu_2+\cdots+\nu_n=n \\ \nu_i\geq 0}}\left(\prod_{k=1}^{n}\nu_k!\right)^{-1}.$$

HINT: No need to write any equation, the two expressions should look familiar. [4 marks]

ANSWER: The summation on the left-hand side is equal to $n!$ by the Cauchy formula (Theorem 72). The summation on the right-hand side is equal to n^n by the multinomial expansion of $(\underbrace{1+1+\cdots+1}_{n})^n$.

9. On your last homework you justified a formula equivalent to

$$Z(S_p^{[2]}) = \frac{1}{p!} \sum_{(j)} \frac{p!}{\prod k^{j_k} j_k!} \prod_{k=1}^{p} s_k^{(k-1)j_k} \prod_{1 \le r < t \le p} s_{[r,t]}^{2(r,t)j_r j_t} \prod_{r=1}^{p} s_r^{r j_r (j_r - 1)}$$

for the number of unlabelled digraphs, where $(r,t) = \gcd(r,t)$, $[r,t] = \mathrm{lcm}(r,t)$ and (j) denotes the set of partitions of p. (a) What is the corresponding formula for the number of unlabelled digraphs where loops are allowed. A *loop* is an edge of the form (v,v). (b) Use the formulas to determine the total number of unlabelled digraphs for $p = 4$, both with loops and without loops. [HINT: Compute $Z(S_p^{[2]}, 2)$ instead of $Z(S_p^{[2]}, 1 + x)$.] [5 marks]

ANSWER: (a) Obviously the vertices of a loop lie in only one cycle. Instead of having $k-1$ cycles of length k, we now have k cycles of length k. Thus the formula is the exactly the same, except that the $(k-1)$ in the product $\prod_{k=1}^{p} s_k^{(k-1)j_k}$ above becomes k.

S_4	no-loops	loops
1^4	s_1^{12}	s_1^{16}
$1^2 2^1$	$6 s_1^2 s_2^5$	$6 s_1^4 s_2^6$
2^2	$3 s_2^6$	$3 s_2^8$
$1^1 3^1$	$8 s_3^4$	$8 s_1^1 s_3^5$
4^1	$6 s_4^3$	$6 s_4^4$

For no-loops we get

$$\frac{1}{24}(2^{12} + 6 \cdot 2^7 + 3 \cdot 2^6 + 8 \cdot 2^4 + 6 \cdot 2^3) = 218.$$

For loops we get

$$\frac{1}{24}(2^{16} + 6 \cdot 2^{10} + 3 \cdot 2^8 + 8 \cdot 2^6 + 6 \cdot 2^4) = 3044.$$

10. A rooted tree is *planted* if it is embedded in the plane. The subtrees incident with the root can rotate around the root, but all other subtrees are ordered. Let p_n denote the number of planted rooted trees with n nodes. [10 marks]

   (a) List the 10 planted rooted trees with 5 nodes.

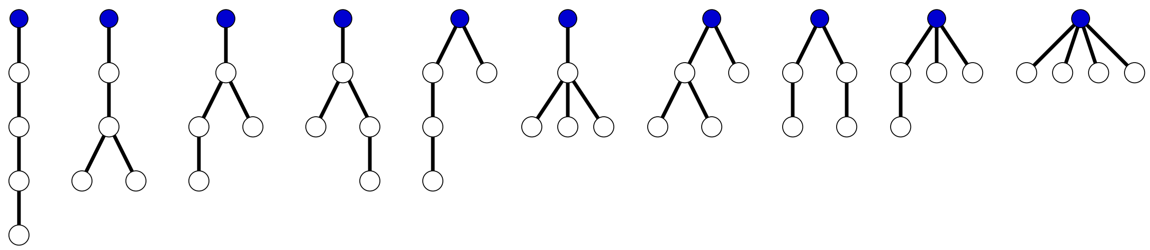
   (b) If C_m is the cyclic group on m elements, what is $Z(C_m)$?

   (c) Let $B(z)$ be the generating function for ordered trees. In terms of $Z(C_m)$ and $B(z)$ what is

$$P(z) = \sum_{n \geq 1} p_n z^n =$$

   (d) Extra Credit: Give a simple (can involve one sum, but no generating functions) expression for p_n?

ANSWER: (a)



(b) If $\sigma = (1\ 2\ \cdots\ m)$, then σ^k consists of $d = \gcd(m,k)$ cycles, each of length m/d. Thus

$$Z(C_m) = \frac{1}{m} \sum_{k=1}^{n} s_{m/\gcd(m,k)}^{\gcd(m,k)} = \frac{1}{m} \sum_{d|m} \phi(d) s_d^{m/d}.$$

(c) A plane rooted tree consists of a single node, or is a root and circular collection of m ordered trees for some $m \geq 1$. Thus

$$P(z) = z + z \sum_{m \geq 1} Z(C_m, B(z)) = z + z \sum_{m \geq 1} \frac{1}{m} \sum_{d|m} \phi(d) B(z^d)^{m/d}.$$

(d) This takes some work, but the eventual answer is

$$p_{n+1} = \frac{1}{2n} \sum_{d|n} \phi(n/d) \binom{2d}{d}.$$