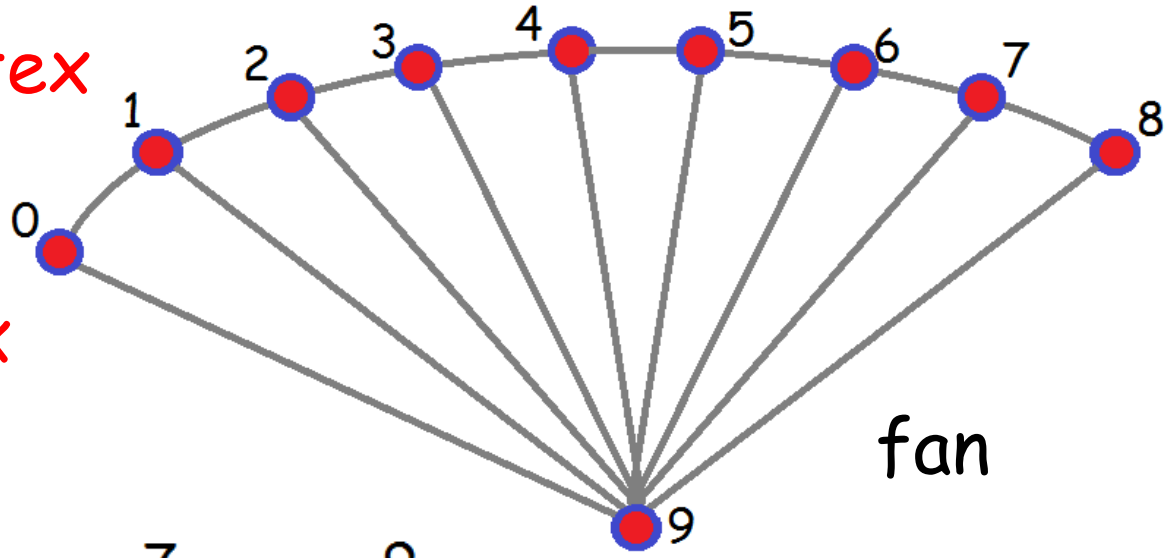
$dc(M') = dc(L) * sc(R) * dc(M) + sc(L) * dc(R) * dc(M)$

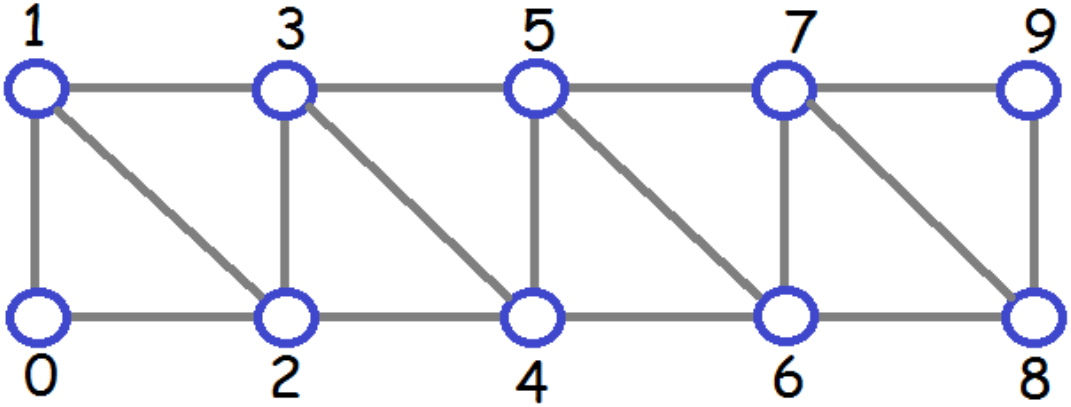
$sc(M') = dc(L) * sc(R) * sc(M) + sc(L) * dc(R) * sc(M)$

$+ sc(L) * sc(R) * dc(M)$

Prove that an n-vertex ladder has the same number of spanning trees as an n-vertex fan.



fan



ladder

Counting Perfect Matchings of a Graph

It is hard in general to count the number of perfect matchings in a graph.

But for planar graphs it can be done in polynomial time.

Pfaffian orientation of a graph:

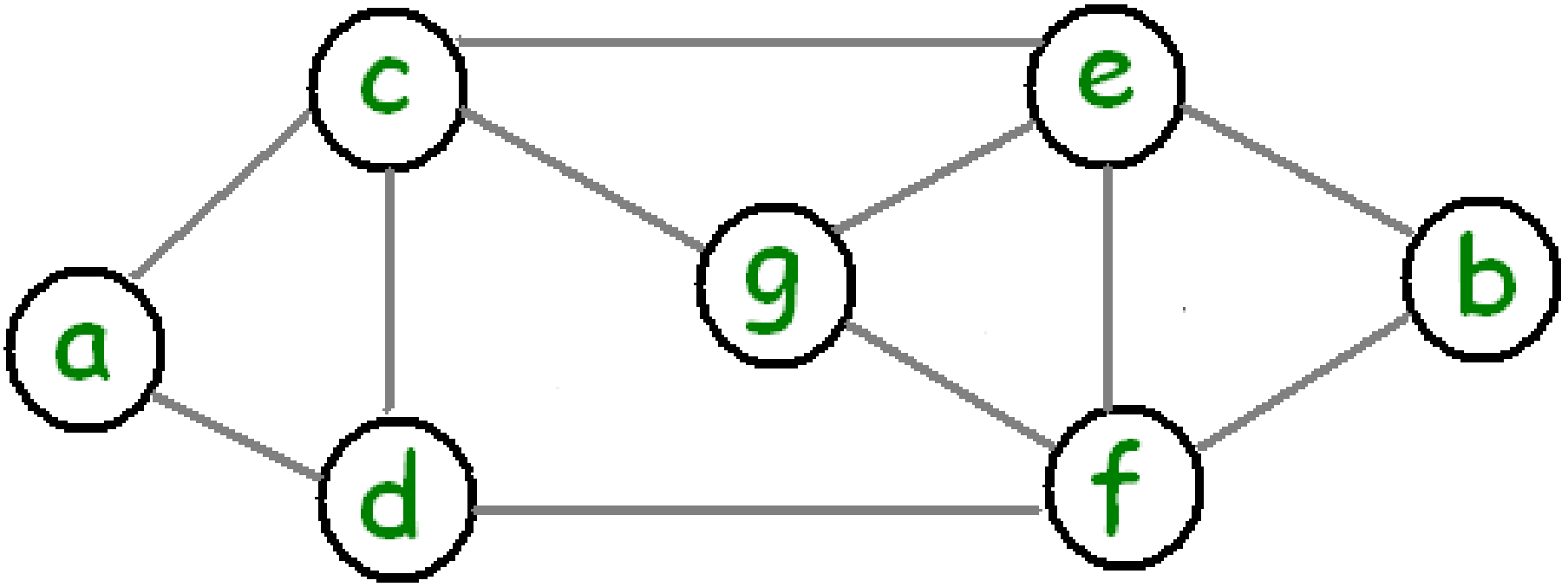
An orientation of a graph G is Pfaffian if every even cycle C such that $G - V(C)$
has a perfect matching has an odd number of edges directed in either direction of the cycle.

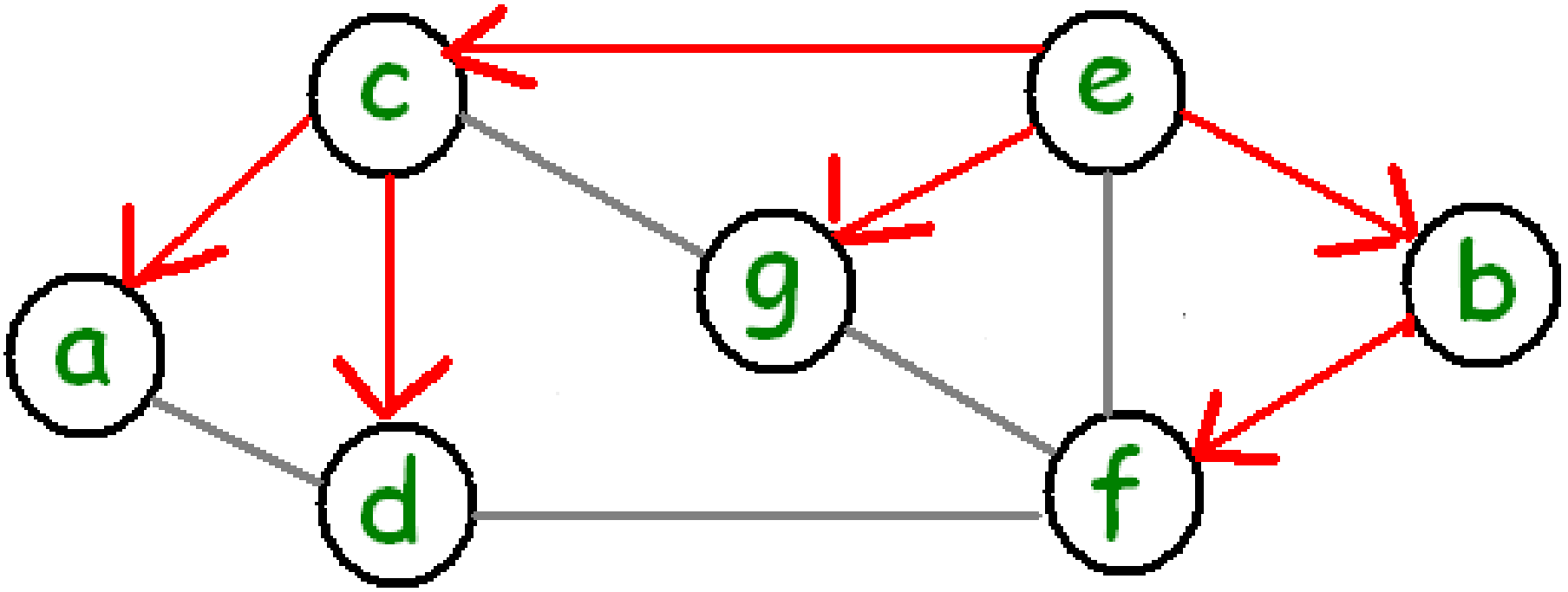
To get a Pfaffian orientation of a planar graph

[Proof of correctness due to Jerrum].

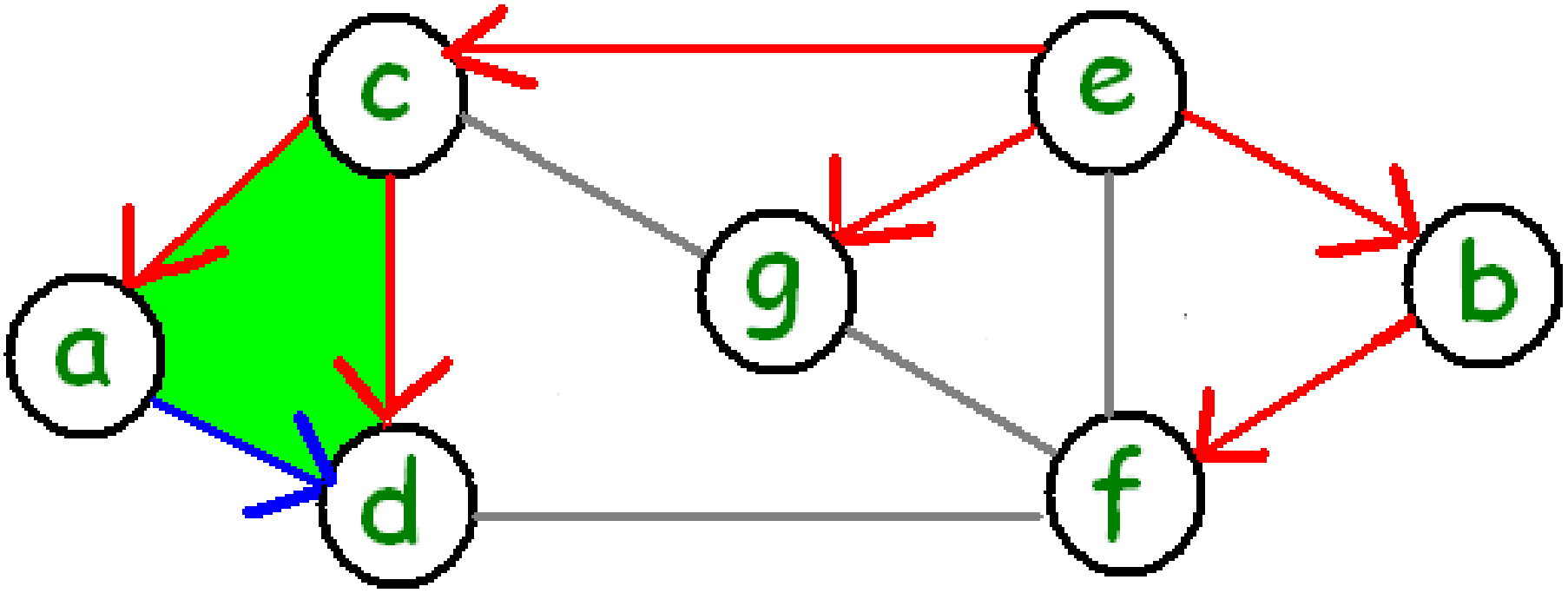
Orient each edge so that all faces except possibly the external one have an odd number of edges going clockwise.

You can always get such an orientation by starting with an oriented tree, then adding edges that finish faces to satisfy this property.


c
e
g
b
a
d
f

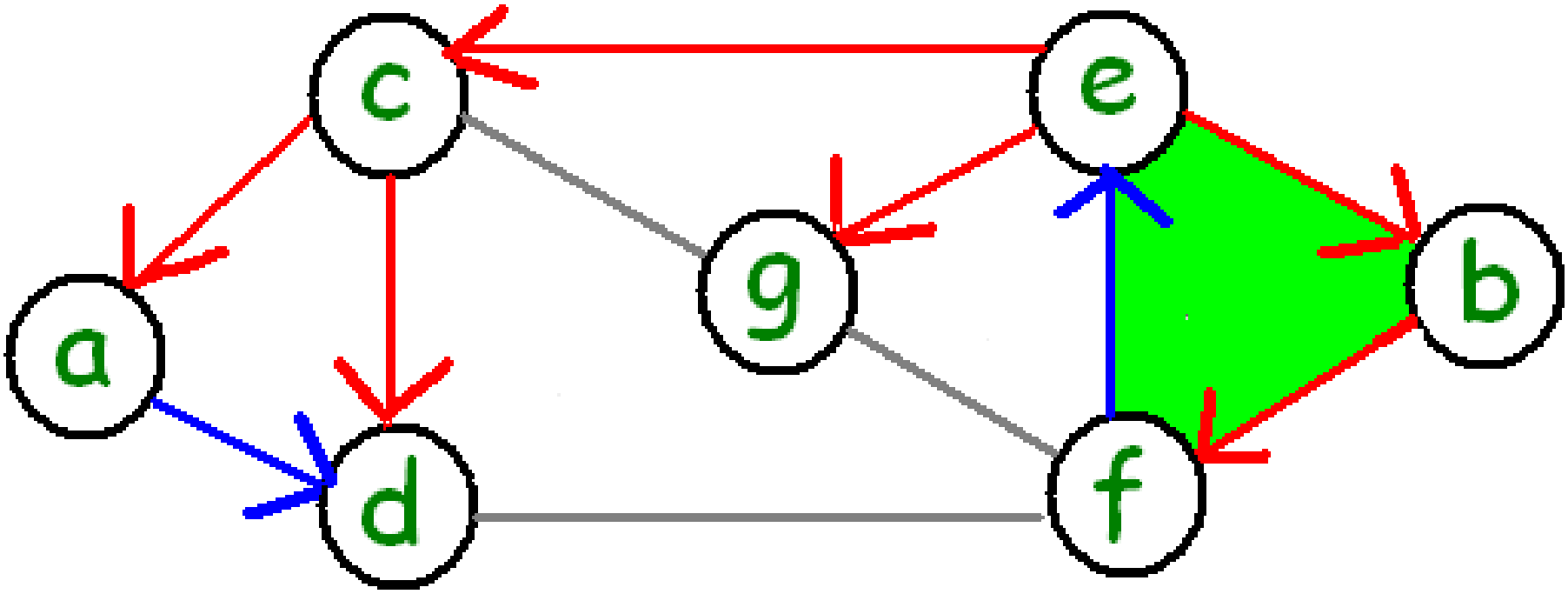


Pick any spanning tree and you can orient those arcs arbitrarily.



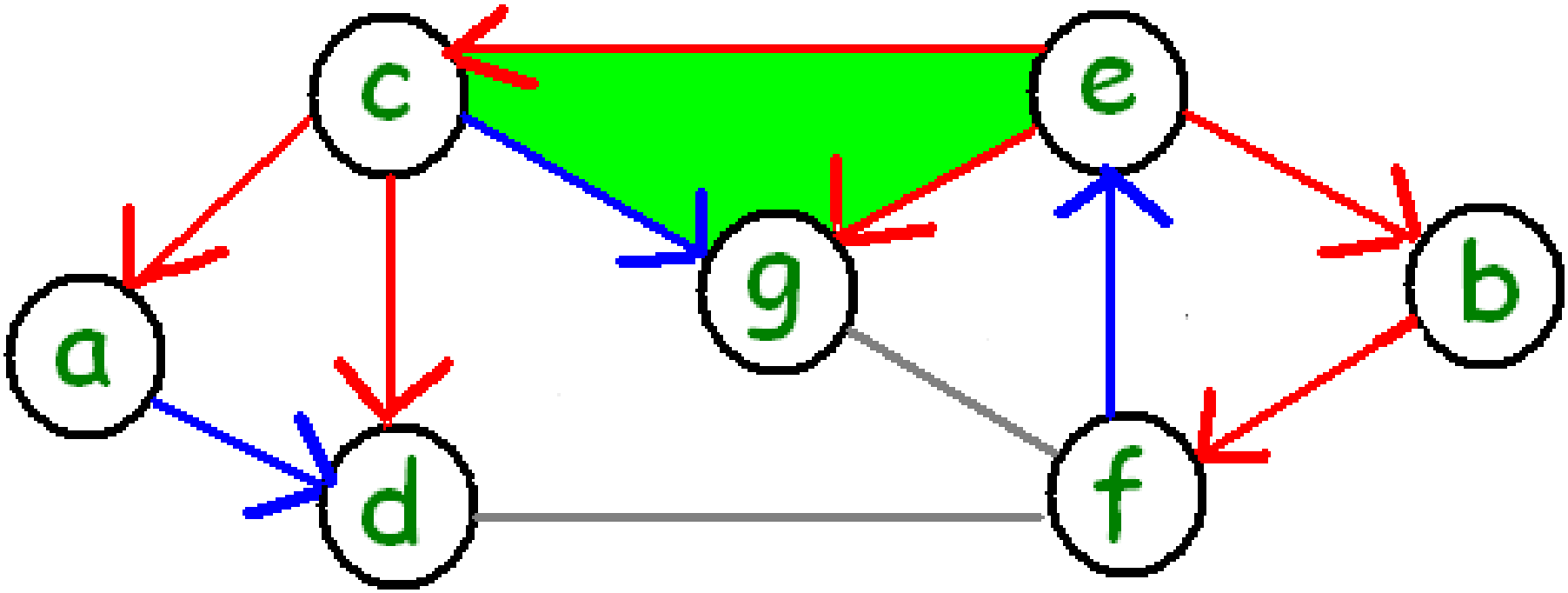
Pick any edge so that it is on some face with all but one of its edges oriented.

Orient it so an odd number of arcs are clockwise. Before 1, so blue arc must go counterclockwise.

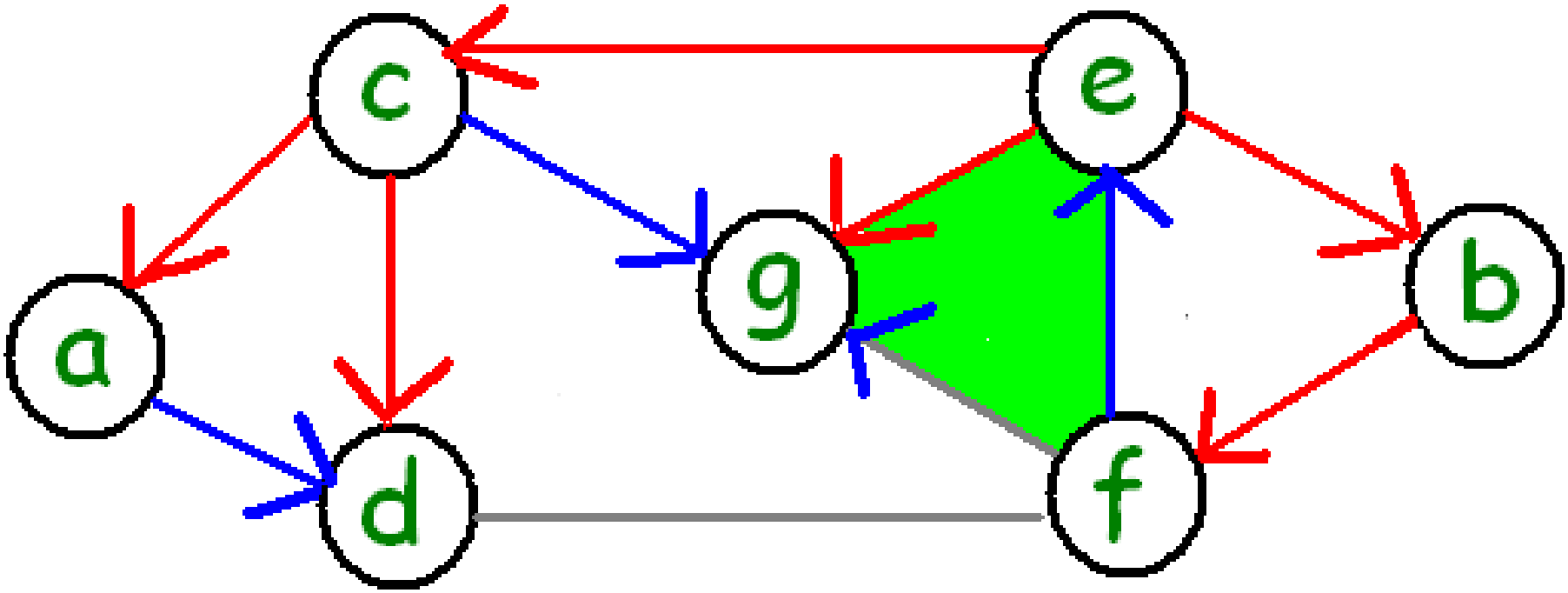


Before 2, so blue arc must go clockwise.

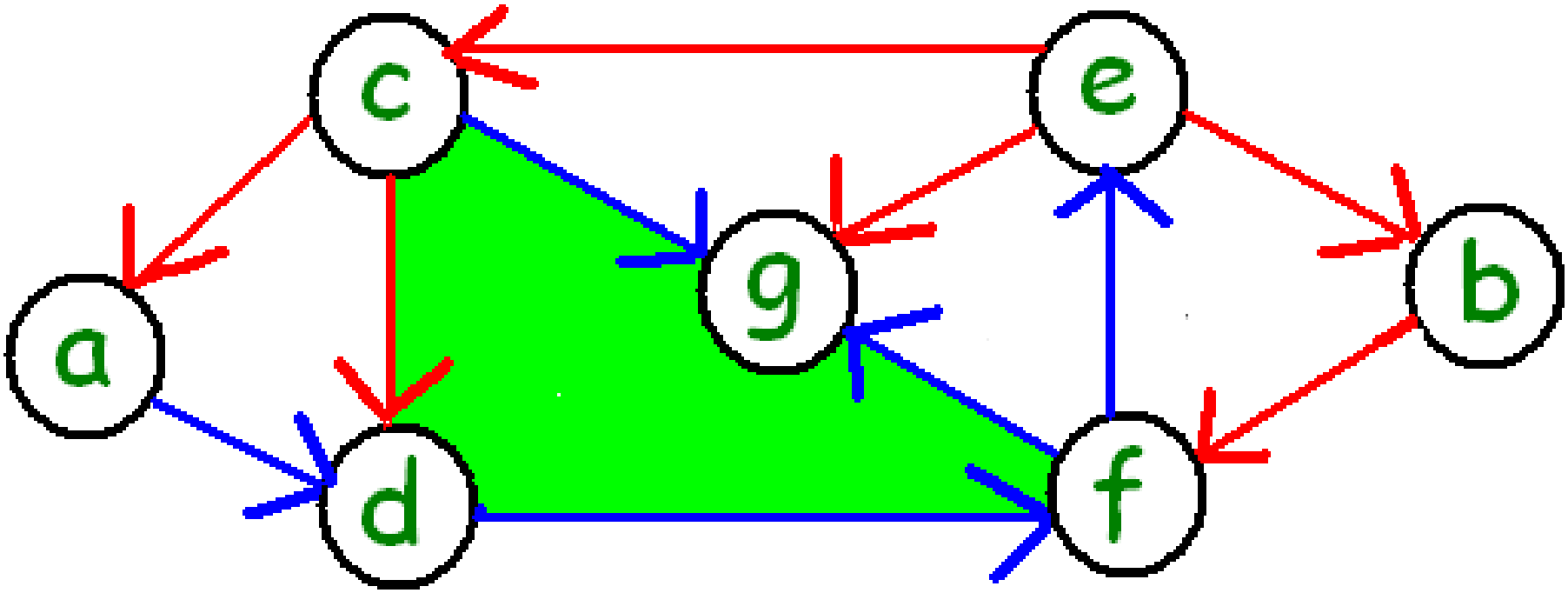
After: 3.



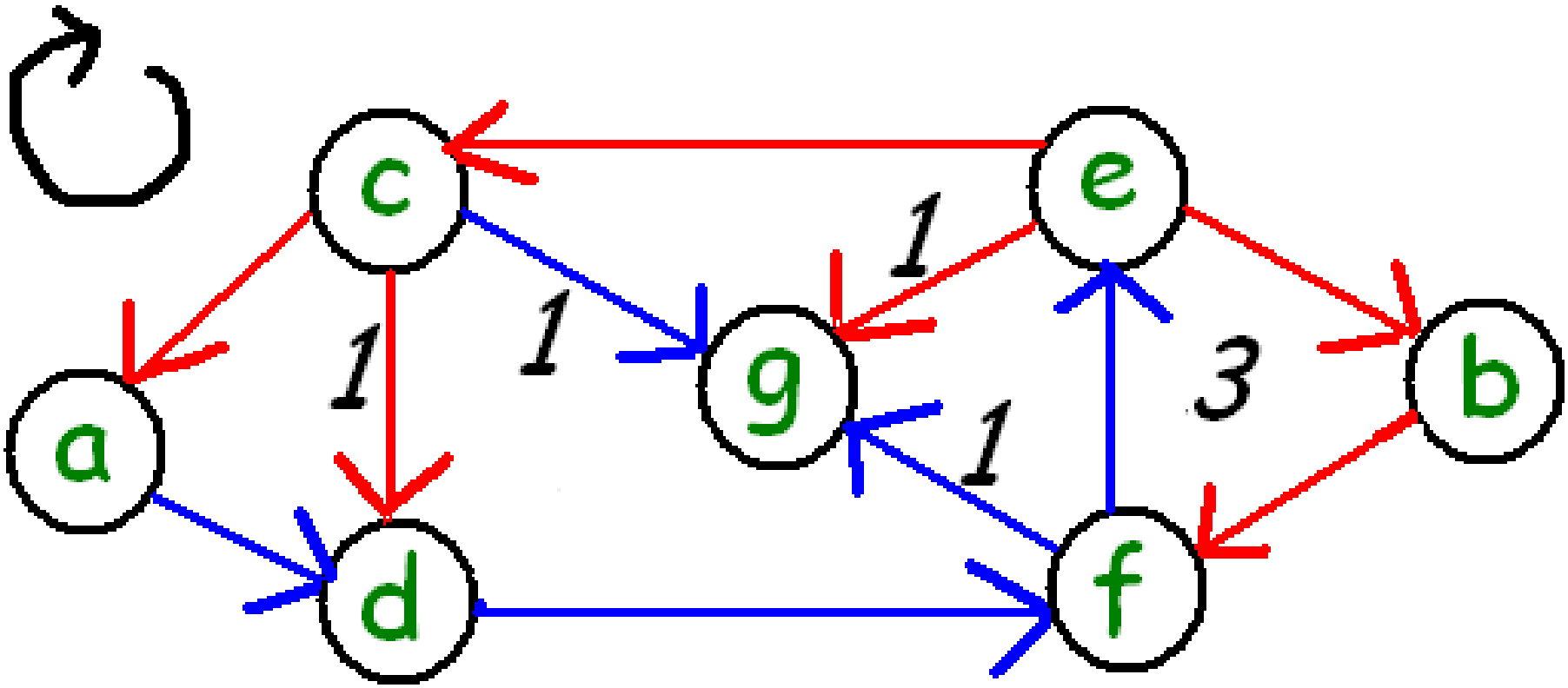
Before 1, so blue arc must go counterclockwise. After: 1.



Before 0, so new blue arc must be (f,g) which is a clockwise arc.

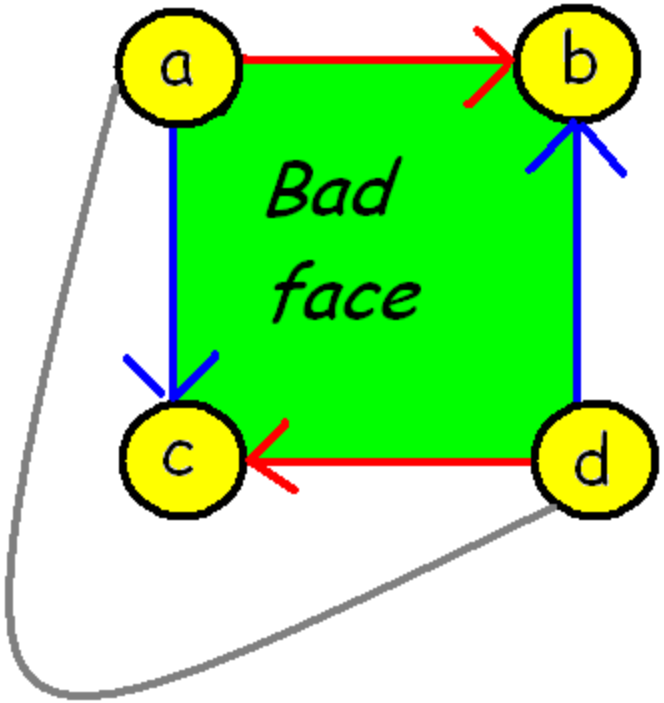


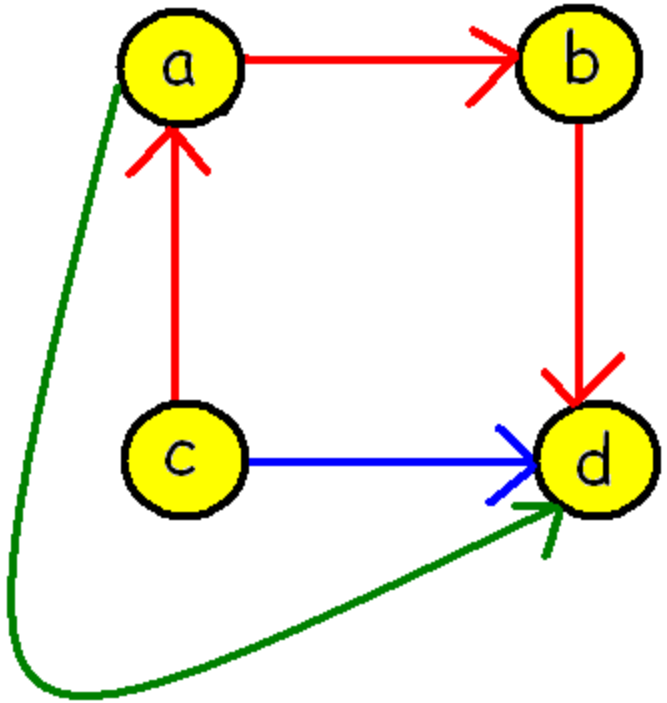
Before 1, so new blue arc must be (d,f) which is a counterclockwise arc.



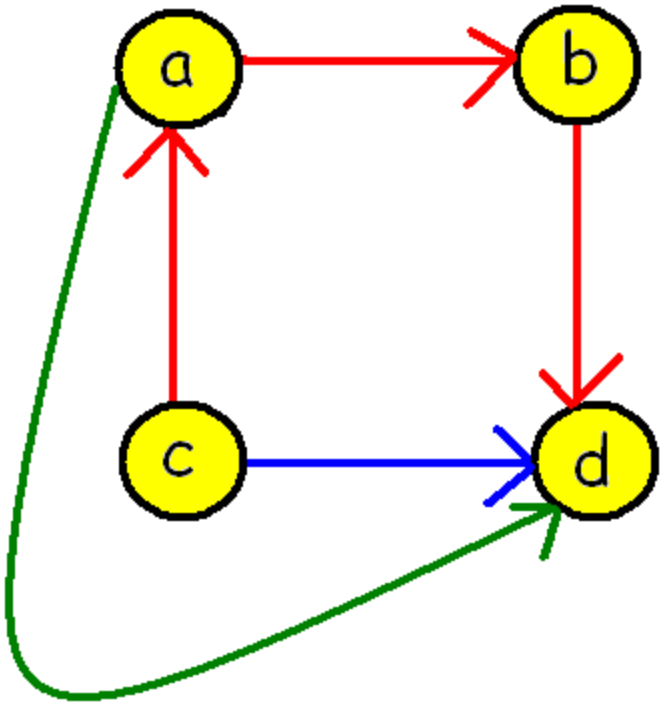
Each internal face has an odd number of clockwise arcs as required.

A smaller example:





Pffafian orientation

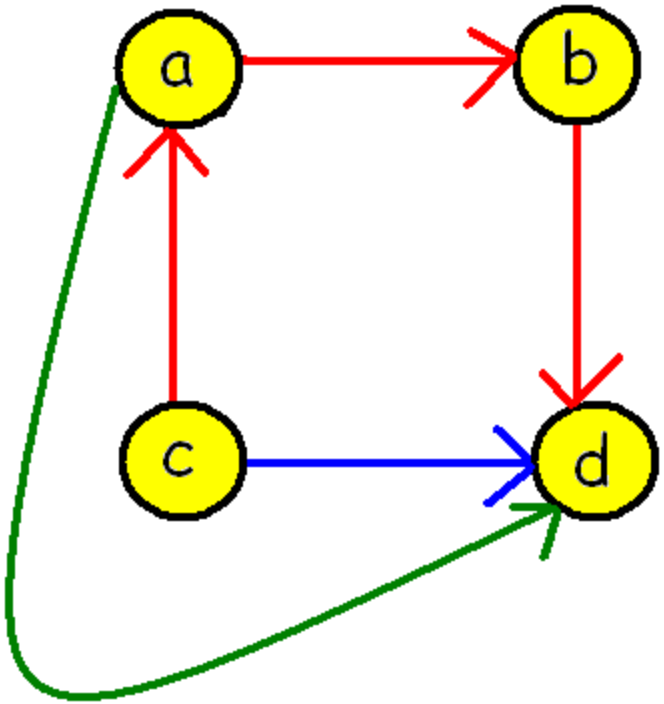


	a	b	c	d
a	0	+1	-1	+1
b	-1	0	0	+1
c	+1	0	0	+1
d	-1	-1	-1	0

If there is an arc (u,v) then position u,v is +1.

If there is an arc (v,u) then position u,v is -1.

If no arc: value is 0.



	a	b	c	d
a	0	+1	-1	+1
b	-1	0	0	+1
c	+1	0	0	+1
d	-1	-1	-1	0

[Number of perfect matchings]^2 = determinant

One way to think of the sign of a permutation:

$\text{sign}(\sigma) = (-1)^s$

where the permutation σ can be sorted (turned into the identity permutation) by using s swap operations.

How many swaps does it take to sort:
1 2 3 0? Cycle structure notation (0123)
How many to sort
1 2 3 4 0? Cycle structure notation (01234)

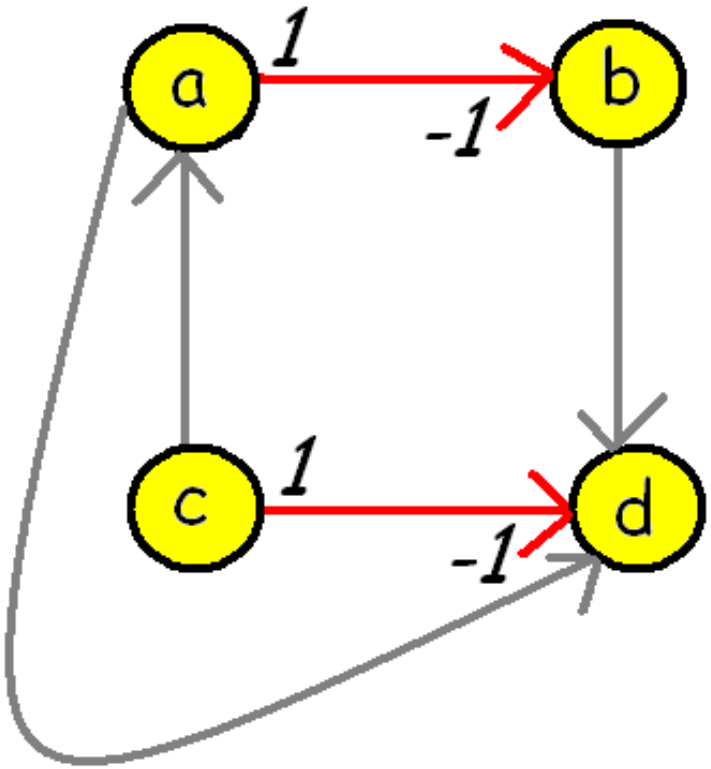
Even cycles contribute -1 to the sign.
Odd cycles contribute +1 to the sign.

One formula for the determinant of an n by n matrix A is: Det(A) =

$$\sum_{\sigma \in S_n} sign(\sigma) \prod_{j=1}^{n} a_{j,\sigma(j)}$$

The sum is computed over all permutations σ of the set {1, 2, ..., *n*}.

Each term of the sum corresponds to a transversal (it has exactly one entry from each row and column of A) of the adjacency matrix.

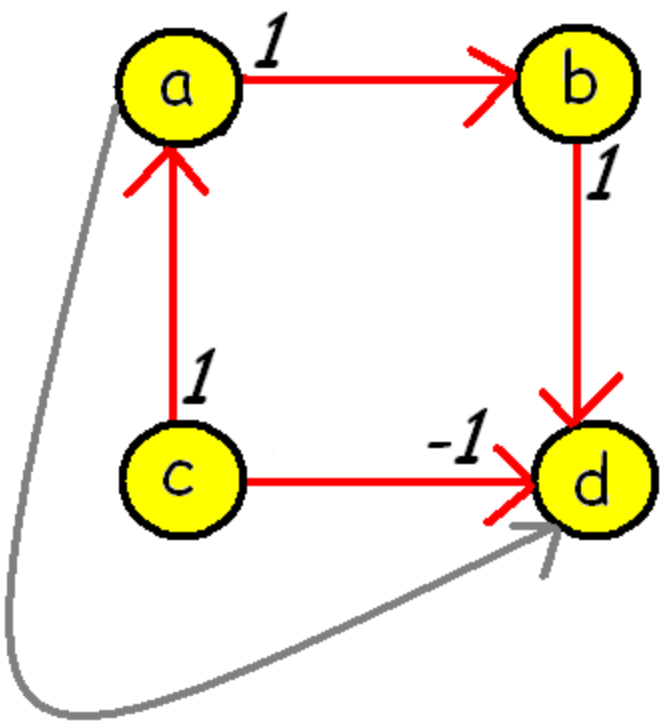


	a	b	c	d
a	0	+1	-1	+1
b	-1	0	0	+1
c	+1	0	0	+1
d	-1	-1	-1	0

Where do non-zero terms of the determinant come from?

Sign term: (a b)(c d) $\Rightarrow$ sign is +1

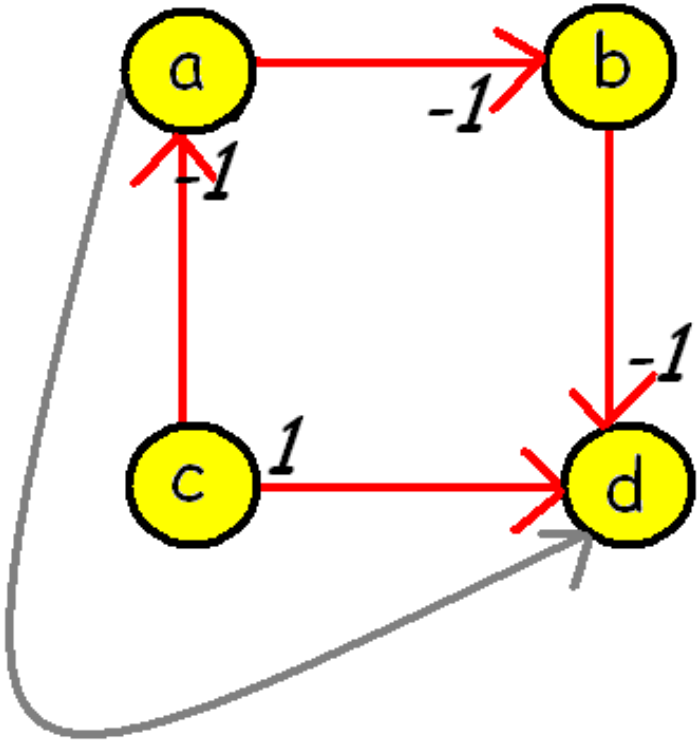
Recall: an even number of even cycles, sign +1.



	a	b	c	d
a	0	+1	-1	+1
b	-1	0	0	+1
c	+1	0	0	+1
d	-1	-1	-1	0

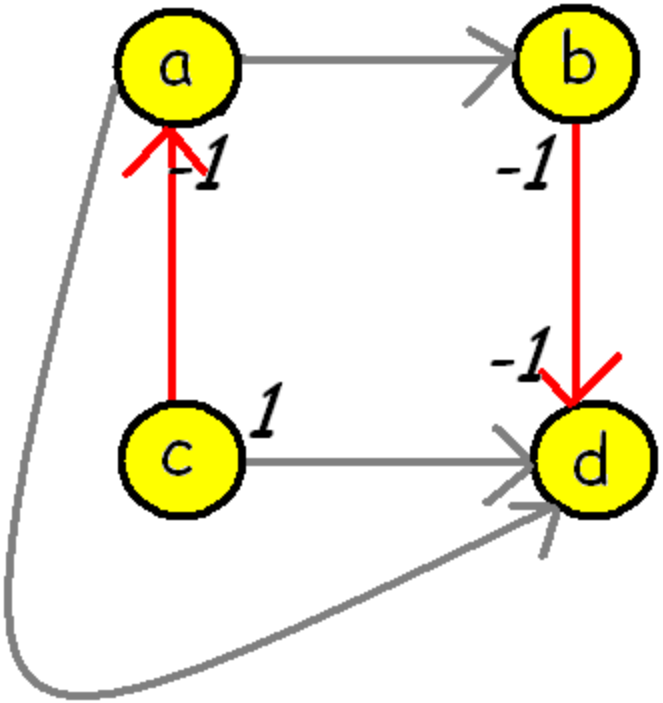
Sign term: (a b d c) $\Rightarrow$ sign is -1

Recall: an odd number of even cycles, sign -1.



	a	b	c	d
a	0	+1	-1	+1
b	-1	0	0	+1
c	+1	0	0	+1
d	-1	-1	-1	0

Sign term: (a c d b) $\Rightarrow$ sign is -1

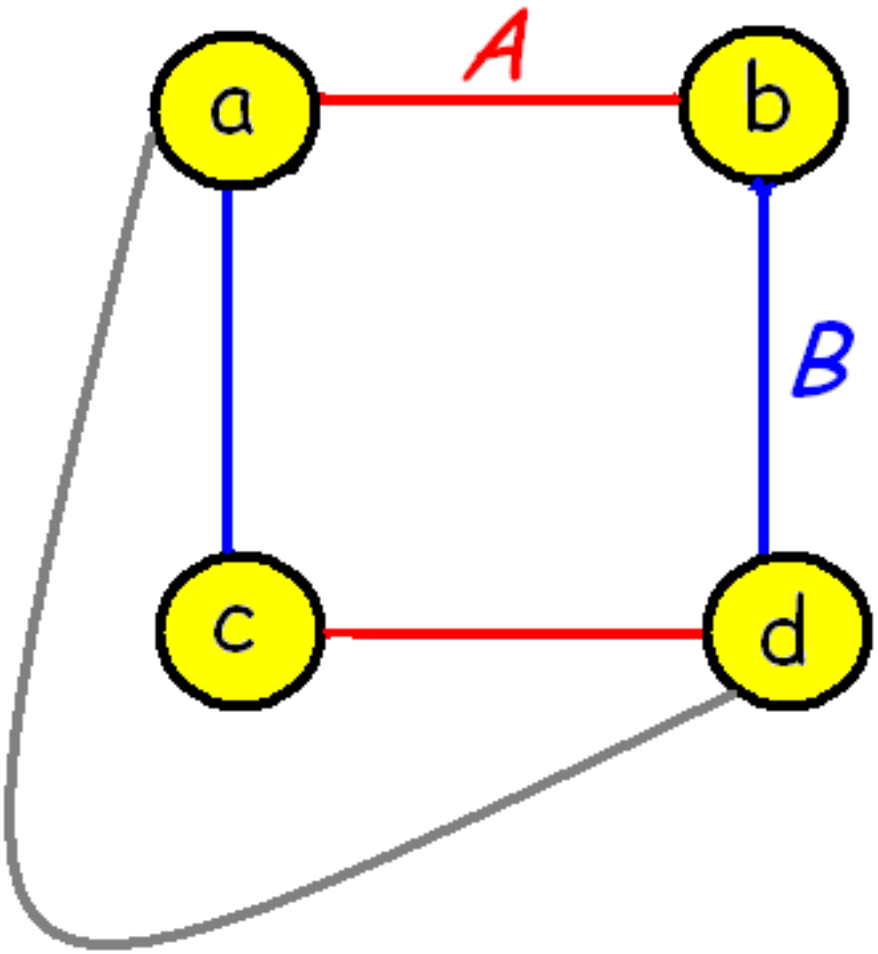


	a	b	c	d
a	0	+1	-1	+1
b	-1	0	0	+1
c	+1	0	0	+1
d	-1	-1	-1	0

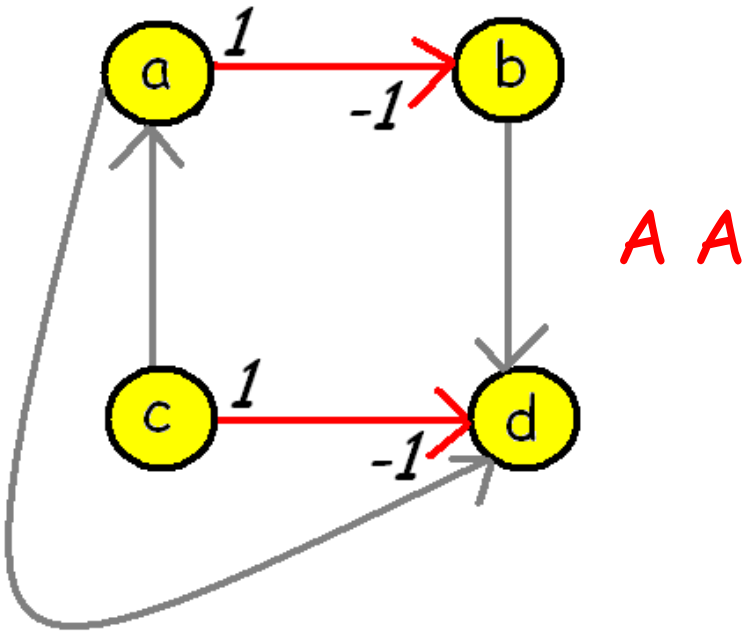
Sign term: (a c)(b d) $\Rightarrow$ sign is -1.

	a	b	c	d
a	0	+1	-1	+1
b	-1	0	0	+1
c	+1	0	0	+1
d	-1	-1	-1	0

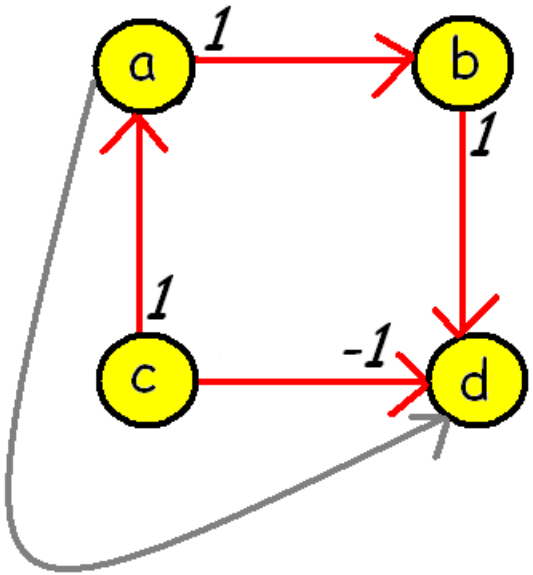
Remaining terms are zero.



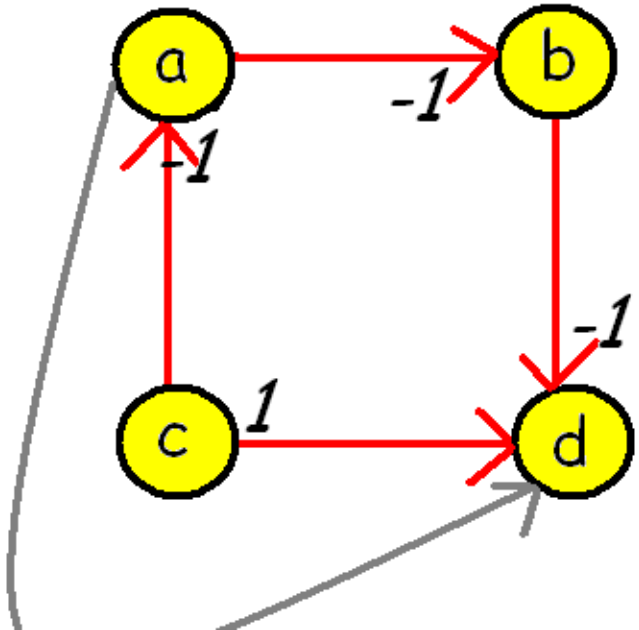
The original graph has two perfect matchings.



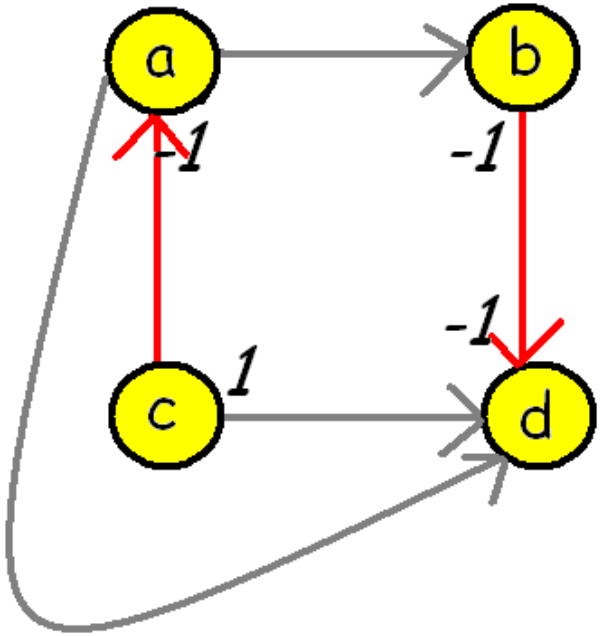
A A



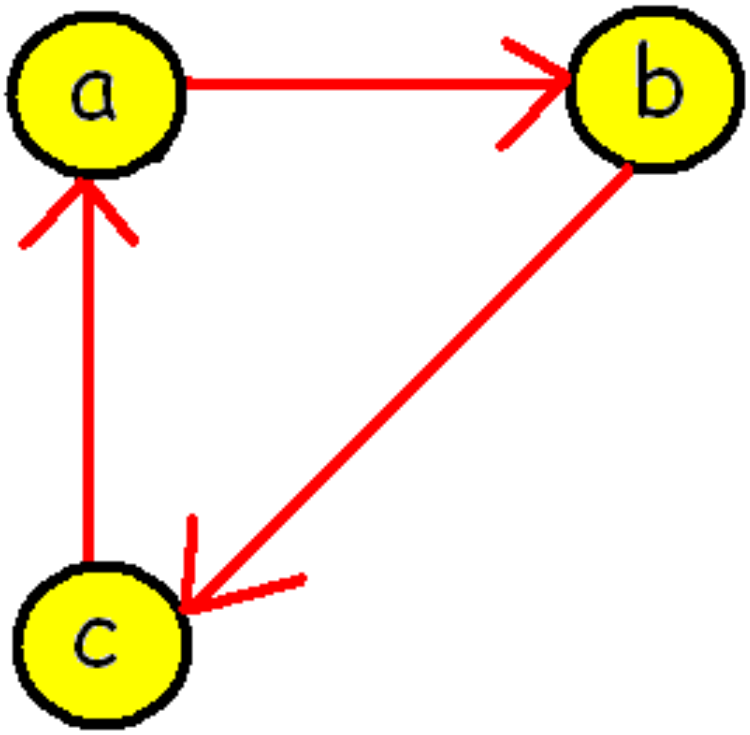
A B



BA

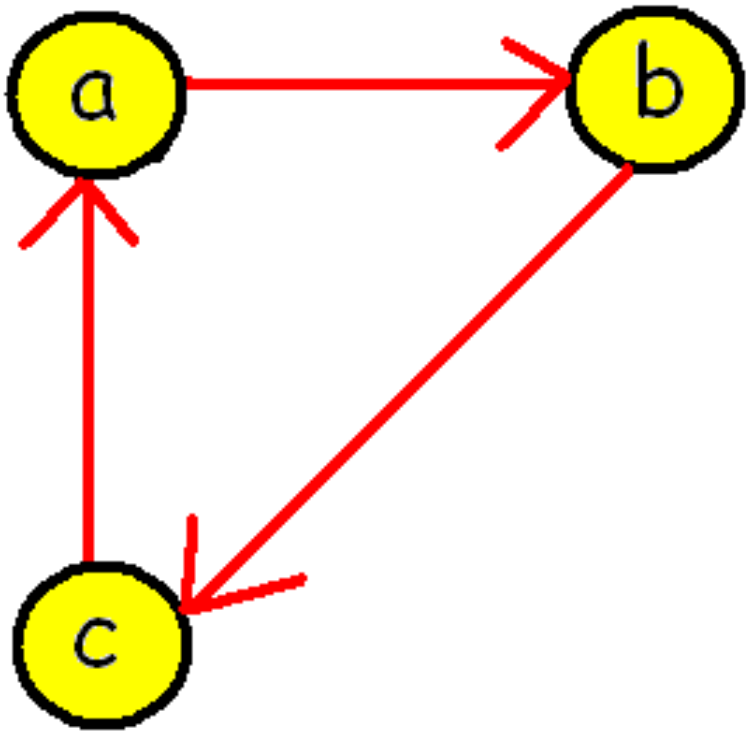


BB



	a	b	c
a	0	1	-1
b	-1	0	1
c	1	-1	0

What happens with odd cycles?



	a	b	c
a	0	1	-1
b	-1	0	1
c	1	-1	0

Non-zero terms:

(a b c) Sign is 1, 1* 1 * 1 * 1 = 1

(a c b) Sign is 1, 1* -1 * -1 * -1 = -1

The non-zero terms of the determinant consist of collections of cycles from the graph.

Even Cycles:

One even cycle has a contribution of:

-1 for the sign term from the determinant,

and it must also have an odd number of arcs going in the traversal direction because even cycles have an odd number of clockwise arcs and hence an odd number of counterclockwise ones and so there is another -1 arising from product of the +1/-1 contributions of its arcs.

Final contribution: -1 * -1 = +1

Odd cycles:

One odd cycle has a contribution of:

+1 for the sign term from the determinant.

Traversing an odd cycle in the two different orders always results in one -1 contribution and one +1 contribution no matter how the edges are directed since if it has an odd number of -1 edges going one way, it must have an even number going the other way.

So the terms come in pairs that cancel each other out!

For each ordered pair (A,B) of matchings has a corresponding term in determinant.

There is a bijection between the ordered pairs of matchings from the graph and the non-zero terms in the determinant that are not the ones cancelled out.

With a Pfaffian orientation, the contributions are all +1 for each collection of even cycles.

This is why the determinant is:

(number of matchings) * (number of matchings)

The eigenvalues of the adjacency matrix A are the values λ such that $Ax = \lambda x$ for some vector x.

The come from solving $\det(A - \lambda I) = 0$.

The characteristic polynomial is a polynomial in λ which is $\det(A - \lambda I)$.

For example: A path with two vertices has its characteristic polynomial equal to $\lambda^2 - 1$.

The coefficients of λ^k in the characteristic polynomial are counting certain subgraphs of the graph in a manner similar to what we just did.

How can a computer scientist compute the determinant of a matrix?

Det(A) =

$$\sum_{\sigma \in S_n} sign(\sigma) \prod_{j=1}^{n} a_{j,\sigma(j)}$$

Applying this formula directly gives an exponential algorithm.

If you multiply a row times a constant c then it multiplies the determinant times c.

Det(A) =

$$\sum_{\sigma \in S_n} sign(\sigma) \prod_{j=1}^{n} a_{j,\sigma(j)}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ ca_{j1} & ca_{j2} & ca_{j3} & & ca_{jn} \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{vmatrix}$$

Each term has one red value so each terms is multiplied by c.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ +caj_1 & +ca_{j2} & +caj_3 & & +caj_n \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ a_{j1} & a_{j2} & a_{j3} & & a_{jn} \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{matrix}$$

Add a constant times a row to another row.

The determinant is the sum of the determinants of these two matrices:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ a_{j1} & a_{j2} & a_{j3} & & a_{jn} \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{vmatrix} \quad \begin{vmatrix} ca_{j1} & ca_{j2} & ca_{j3} & \dots & ca_{jn} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ a_{j1} & a_{j2} & a_{j3} & & a_{jn} \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{vmatrix}$$

But the determinant of the second one is 0 because it has two rows that are constant multiples of each other. So the determinant does not change.

Swapping two adjacent rows: multiplies the determinant by -1.

So: apply Gaussian elimination to make the matrix upper triangular keeping track of what you have done to the determinant at each step.

The determinant of an upper triangular matrix is the product of its diagonal entries.

$$\begin{vmatrix} a_{11} & 0 & 0 & & \dots & 0 \\ a_{21} & a_{22} & 0 & & \dots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \dots & 0 \\ & & & & & 0 \\ a_{n1} & a_{n2} & a_{n3} & & & a_{nn} \end{vmatrix}$$